ACTA UNIVERSITATIS SZEGEDIENSIS

ACTA SCIENTIARUM MATHEMATICARUM

TOMUS 47

FASC. 1—2

SZEGED, 1984

INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

ACTA SCIENTIARUM MATHEMATICARUM

47. KÖTET

FASC. 1—2.

SZEGED, 1984

JÓZSEF ATTILA TUDOMÁNYEGYETEM BOLYAI INTÉZETE

ACTA UNIVERSITATIS SZEGEDIENSIS

ACTA SCIENTIARUM MATHEMATICARUM

TOMUS 47

SZEGED, 1984

INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

ACTA SCIENTIARUM MATHEMATICARUM

47. KÖTET
FASC. 1—2

SZEGED, 1984

JÓZSEF ATTILA TUDOMÁNYEGYETEM BOLYAI INTÉZETE

Acta Sci. Math., 47 (1984), 3—39

Topological quasi varieties

DAVID M. CLARK* and PETER H. KRAUSS

A *quasi variety* is a class of *algebras* defined by a set of *quasi equations*

$$\bigwedge_{i<n} \Phi_i \Rightarrow \Psi,$$

where each Φ_i and Ψ are *equations*. This well known notion can be generalized in many ways to accommodate the need for more powerful means of expression. The notion studied in this paper encompasses two such generalizations going into different directions. In one direction we follow GRÄTZER and LAKSER [11] who introduce *structures* with *operations* and *relations*. They then generalize a result of MAL'CEV [17] to this setting: A class of structures is a quasi variety if and only if it is closed under the formation of isomorphic images, substructures and reduced products. Continuing in this direction ANDRÉKA, BURMEISTER and NÉMETI [1] consider *partial algebras* and prove the corresponding result. In another direction we follow TAYLOR [22] who considers *topological algebras* and introduces a new type of (infinitary) *topological atomic formula* to express *net convergence*. He then generalizes a result of BIRKHOFF [3] to this setting: A class of topological algebras is definable by topological atomic formulas if and only if it is closed under the formation of continuous homomorphic images, subalgebras and direct products. In this paper we shall consider *topological structures* (with operations, partial operations, relations and a topology) and introduce topological atomic formulas to talk about the topology. Since these new atomic formulas are infinitary, we have to allow for infinite conjunctions in *topological quasi atomic formulas*

$$\bigwedge\{\Phi_i | i\in I\} \Rightarrow \Psi,$$

where each Φ_i and Ψ are atomic formulas and I is a *set*. A *topological quasi variety* is a class of topological structures defined by a class of topological quasi atomic

Received November 18, 1981.

* Research supported by the SUNY Research Foundation and the Alexander von Humboldt Foundation.

formulas. In this setting we show that a class of topological structures is a topological quasi variety if and only if it is closed under the formation of topological isomorphic images, substructures and direct products.

The definition of this notion was motivated by the observation that topological quasi varieties naturally arise in the numerous topological duality theorems in the literature. In each case a category anti equivalence is established between the quasi variety generated by a finite algebra $\mathfrak{P}$ and the class of *compact* members of the topological quasi variety generated by a finite topological structure $\boldsymbol{P}$ having the same universe as $\mathfrak{P}$. Accordingly we call the class of compact members of a topological quasi variety a *compact topological quasi variety*. It easily follows that a class of compact topological structures is a compact topological quasi variety if and only if it is closed under the formation of topological isomorphic images, compact substructures and direct products.

Duality theory is a rather recent topic in universal algebra and the primary sources are DAVEY [6] and DAVEY and WERNER [7]. We are mostly interested in their notion of *full duality* and we review the conceptual framework in which this notion is introduced. We then give a new characterization of this notion in terms of *hull-kernel closed* sets which not only tends to clarify the situation but also leads to new duality results. We show that there is a full duality for the (quasi) variety generated by an arbitrary finite algebra having a *near unanimity term* and only *simple non-trivial subalgebras*. This includes all *dual discriminator algebras* and unifies previous results for quasi primal algebras, distributive lattices, weakly associative lattices, median algebras, Kleene algebras and DeMorgan algebras. The full duality results for quasi primal algebras, weakly associative lattices and median algebras were claimed by WERNER [23] and DAVEY and WERNER [7], however their proofs are not correct. So we not only vindicate their claims but also establish them in a much broader context.

Topological quasi atomical theories appear to be an interesting topic for the logician and — as far as we have been able to ascertain — some of the most obvious problems in this area are still open. We have addressed ourselves to the question of *axiomatizability* and have come up with axiomatizations of several topological quasi atomical theories, many of which arise in the context of duality theory.

Altogether, we have attempted to put some rather diverse but extremely interesting recent developments in universal algebra and model theory into a unifying perspective.

1. Topological quasi varieties

In this section we define topological quasi varieties and investigate their basic properties. Special features are a suitable treatment of partial operations and the introduction of an infinitary first-order language which permits us to express many relevant topological facts.

Given is a similarity type $\mathbf{t}$ determined by a set $\mathbf{Op}$ of *operation symbols*, a set $\mathbf{POp}$ of *partial operation symbols* and a set $\mathbf{Rl}$ of *relation symbols*. A *topological structure* $\mathbf{X}$ of similarity type $\mathbf{t}$ has a (non-empty) topological space X for a universe, for each n-ary operation symbol $f \in \mathbf{Op}$ has a continuous n-ary operation $f^{\mathbf{X}}: X^n \to X$, for each n-ary partial operation symbol $g \in \mathbf{POp}$ has a continuous n-ary partial operation $g^{\mathbf{X}}: D \to X$, where $D = \mathrm{dom}\,(g^{\mathbf{X}}) \subseteq X^n$ is the (possibly empty) domain of $g^{\mathbf{X}}$, and for each n-ary relation symbol $r \in \mathbf{Rl}$ has a closed n-ary relation $r^{\mathbf{X}} \subseteq X^n$, where X^n is endowed with the product topology. In all constructions involving topological structures operations and relations behave as usual so that we shall only mention them in case something extraordinary or unexpected is happening. Although all topological constructions are standard as well, we shall be a little more explicit in this area because there are some subtleties which are easily overlooked. The situation is quite different with partial operations. There are several options available here which have been pursued in the literature (see, e.g., GRÄTZER [10]). Thus we have to make our choices quite explicit in this area. $\mathbf{X}$ is called an *algebra* in case it has neither partial operations nor relations, i.e. $\mathbf{POp} \cup \mathbf{Rl} = \emptyset$.

To begin with, for $\mathbf{Y}$ to be a *substructure* of $\mathbf{X}$ (in symbols $\mathbf{Y} \subseteq \mathbf{X}$) we require that Y is a subspace of X and for all $g \in \mathbf{POp}$, $g^{\mathbf{Y}} \subseteq g^{\mathbf{X}}$ and $\mathrm{dom}\,(g^{\mathbf{Y}}) = \mathrm{dom}\,(g^{\mathbf{X}}) \cap Y^n$. For φ to be a *continuous homomorphism* from $\mathbf{X}$ to $\mathbf{Y}$ (in symbols $\varphi: \mathbf{X} \to \mathbf{Y}$) we require that φ is continuous and for all $g \in \mathbf{POp}$, if $x \in \mathrm{dom}\,(g^{\mathbf{X}})$, then $\varphi x \in \mathrm{dom}\,(g^{\mathbf{Y}})$ and $\varphi g^{\mathbf{X}}(x) = g^{\mathbf{Y}}(\varphi x)$. In the presence of partial operations and/or relations homomorphisms are afflicted with some peculiarities which cause much trouble and confusion in this area. To be more specific, if $\varphi: \mathbf{X} \to \mathbf{Y}$, then the following conditions are not necessarily satisfied:

(1) If $g \in \mathbf{POp}$ and $\varphi x \in \mathrm{dom}\,(g^{\mathbf{Y}})$, then there exists $y \in \mathrm{dom}\,(g^{\mathbf{X}})$ such that $\varphi x = \varphi y$.

(2) If $r \in \mathbf{Rl}$ and $\varphi x \in r^{\mathbf{Y}}$, then there exists $y \in r^{\mathbf{X}}$ such that $\varphi x = \varphi y$.

Notice that (1) implies that $\varphi(X) \subseteq Y$ is closed under $g^{\mathbf{Y}}$, i.e. $\varphi(X)$ determines a substructure of $\mathbf{Y}$, but not vice versa. The fact that $\varphi(X)$ may fail to determine a substructure of $\mathbf{Y}$ will be an important issue later. This situation seriously affects the definition of *surjective homomorphism* and *homomorphic image*, and there are several options available for these notions. The notion of *injective homomorphism* is not entirely clear either. Fortunately we do not have to get involved in these troublesome decisions because we only have to consider *embeddings* where the requirements are quite clear. Consequently we shall use the notions "injective" and "surjective" *only for mappings.* φ is called a *(topological) embedding* (in symbols $\varphi\colon \mathbf{X} \Vert\!\!\rightarrow \mathbf{Y}$) in case φ is a continuous homomorphism and an injective mapping whose inverse is continuous, where

(i) for all $g \in \mathbf{POp}$, $x \in \operatorname{dom}(g^{\mathbf{X}})$ iff $\varphi x \in \operatorname{dom}(g^{\mathbf{Y}})$;

(ii) for all $r \in \mathbf{Rl}$, $x \in r^{\mathbf{X}}$ iff $\varphi x \in r^{\mathbf{Y}}$.

φ is called a *topological isomorphism* (in symbols $\varphi\colon \mathbf{X} \Vert\!\!\twoheadrightarrow \mathbf{Y}$) if, in addition, φ is a surjective mapping (equivalently, if φ has a two-sided inverse). Now it is clear that a topological embedding is a topological isomorphism with a substructure. For the *direct product* $\mathbf{Y} = \prod \langle \mathbf{X}_i \mid i \in I \rangle$ of the set of structures $\{\mathbf{X}_i \mid i \in I\}$ we require that Y is endowed with the product topology and that for all $g \in \mathbf{POp}$ and all $y \in Y^n$,

$$y \in \operatorname{dom}(g^{\mathbf{Y}}) \quad \text{iff} \quad y(i) \in \operatorname{dom}(g^{\mathbf{X}_i}) \quad \text{for all} \quad i \in I, \quad \text{and}$$

$$g^{\mathbf{Y}}(y_0, \ldots, y_{n-1})(i) = g^{\mathbf{X}_i}(y_0(i), \ldots, y_{n-1}(i)).$$

By a *trivial structure* we mean a one element structure with all partial operations and relations nonempty. For example, the direct product $\prod \langle \mathbf{X}_i \mid i \in \emptyset \rangle$ of the empty set of structures is trivial.

If $J \subseteqq I$, then the projection

$$\pi_J\colon \mathbf{Y} \rightarrow \prod \langle \mathbf{X}_j | j \in J \rangle$$

is afflicted with both peculiarities mentioned above, i.e. (1) and (2) both may fail although π_J clearly is a *surjective mapping.*

If $\mathscr{K}$ is a class of topological structures then $\mathbf{I}\mathscr{K}$, $\mathbf{S}\mathscr{K}$ and $\mathbf{S}_c\mathscr{K}$ denote the classes of topological isomorphic images, substructures and compact substructures of members of $\mathscr{K}$ respectively, and $\mathbf{P}\mathscr{K}$ denotes the class of direct products of *nonempty* subsets of $\mathscr{K}$. First we show that our definition of direct product is correct for the category we have defined.

Lemma 1.1. (Transfer Principle). *Suppose $I \neq \emptyset$ and for each $i \in I$, $\varphi_i\colon \mathbf{Y} \rightarrow \mathbf{X}_i$. Then there exists a unique $\psi\colon \mathbf{Y} \rightarrow \prod \langle \mathbf{X}_i \mid i \in I \rangle$ such that for all $i \in I$, $\varphi_i = \pi_i \circ \psi$.*

Proof. Suppose $g \in \mathbf{POp}$ and $y \in \operatorname{dom}(g^{\mathbf{Y}})$. By hypothesis, for every $i \in I$, $\varphi_i y \in \operatorname{dom}(g^{\mathbf{X}_i})$ and $\varphi_i g^{\mathbf{Y}}(y) = g^{\mathbf{X}_i}(\varphi_i y)$. Thus $\psi y(i) \in \operatorname{dom}(g^{\mathbf{X}_i})$ for all $i \in I$ and therefore $\psi y \in \operatorname{dom}(g^{\mathbf{Z}})$, where $\mathbf{Z} = \prod \langle \mathbf{X}_i \mid i \in I \rangle$. Finally, $\psi g^{\mathbf{Y}}(y)(i) = \varphi_i g^{\mathbf{Y}}(y) = g^{\mathbf{X}_i}(\varphi_i y) = g^{\mathbf{X}_i}(\psi y(i)) = g^{\mathbf{Z}}(\psi y)(i)$ for all $i \in I$.

Next we show that the usual separation properties can be augmented to obtain subdirect representations in this setting.

Lemma 1.2 (Separation Principle). *For any nontrivial structure $\mathbf{Y}$, $\mathbf{Y} \in \mathbf{ISP}\mathscr{K}$ if and only if*

(i) *If* $x, y \in Y$, *where* $x \neq y$, *then there exist* $\mathbf{X} \in \mathscr{K}$ *and* $\varphi\colon \mathbf{Y} \to \mathbf{X}$ *such that* $\varphi x \neq \varphi y$.

(ii) *If* $g \in \mathbf{POp}$ *and* $y \notin \operatorname{dom}(g^{\mathbf{Y}})$, *then there exist* $\mathbf{X} \in \mathscr{K}$ *and* $\varphi\colon \mathbf{Y} \to \mathbf{X}$ *such that* $\varphi y \notin \operatorname{dom}(g^{\mathbf{X}})$.

(iii) *If* $r \in \mathbf{Rl}$ *and* $y \notin r^{\mathbf{Y}}$, *then there exist* $\mathbf{X} \in \mathscr{K}$ *and* $\varphi\colon \mathbf{Y} \to \mathbf{X}$ *such that* $\varphi y \notin r^{\mathbf{X}}$.

(iv) *If D is a subset of the power set of Y directed by inclusion,* $\delta\colon D \to Y$ *is a net in Y and* $y \in Y$, *where* $\delta d \overset{d\in D}{\nrightarrow} y$, *then there exist* $\mathbf{X} \in \mathscr{K}$ *and* $\varphi\colon \mathbf{Y} \to \mathbf{X}$ *such that* $\varphi \circ \delta d \overset{d\in D}{\nrightarrow} \varphi y$.

Proof. The necessity of (i)—(iv) is obvious. So assume (i)—(iv). We consider all instances mentioned in conditions (i)—(iv), and for each instance we choose $\mathbf{X} \in \mathscr{K}$ and $\varphi\colon \mathbf{Y} \to \mathbf{X}$ "witnessing" its occurrence. So let $\mathbf{X}_i \in \mathscr{K}$, $\varphi_i\colon \mathbf{Y} \to \mathbf{X}_i$, where $i \in I$, be a set of "witnesses". By the Transfer Principle, there exists unique

$$\psi\colon \mathbf{Y} \to \prod \langle \mathbf{X}_i \mid i \in I\rangle$$

such that for all $i \in I$, $\varphi_i = \pi_i \circ \psi$. We shall show that ψ is a topological embedding. As usual, the witnesses for (i) make ψ an injective mapping. To show that ψ^{-1} is continuous it suffices to establish that ψ is a closed mapping. So suppose $W \subseteq Y$, where $\psi(W) \subseteq \psi(Y)$ is not closed. Then there exists $y \in Y$ such that ψy is a limit point of $\psi(W)$ but $\psi(y) \notin \psi(W)$. Let D be the set of $U \subseteq Y$ such that $\psi(U)$ is a neighborhood of ψy. D is directed by inclusion. For each $U \in D$ there exists $\delta U \in Y$ such that $\psi \circ \delta U \in \psi(U) \cap \psi(W)$. Thus $\psi \circ \delta U \overset{U\in D}{\to} \psi y$. If $\delta U \overset{U\in D}{\nrightarrow} Y$, then by construction we obtain from (iv) $\mathbf{X}_i \in \mathscr{K}$ and $\varphi_i\colon \mathbf{Y} \to \mathbf{X}_i$ such that $\varphi_i \circ \delta U \overset{U\in D}{\nrightarrow} \varphi_i y$ and therefore $\pi_i \circ \psi \circ \delta U \overset{U\in D}{\nrightarrow} \pi_i \circ \psi y$, contradicting net convergence in direct products. Thus $\delta U \overset{U\in D}{\to} y$ so that y is a limit point of W, but $y \notin W$. It follows that W is not closed so that ψ^{-1} is continuous.

Next, suppose $g \in \mathbf{POp}$ and $y \notin \operatorname{dom}(g^{\mathbf{Y}})$. Then by construction we obtain from (ii) $\mathbf{X}_i \in \mathscr{K}$ and $\varphi_i\colon \mathbf{Y} \to \mathbf{X}_i$ such that $\varphi_i y \notin \operatorname{dom}(g^{\mathbf{X}_i})$. Thus $\psi y(i) \notin \operatorname{dom}(g^{\mathbf{X}_i})$ and therefore $\psi y \notin \operatorname{dom}(g^{\mathbf{Z}})$, where $\mathbf{Z} = \prod \langle \mathbf{X}_i \mid i \in I\rangle$.

Since the witnesses for (iii) take care of the relations, ψ is indeed a topological embedding.

The Separation Principle considerably simplifies for compact Hausdorff structures.

Corollary 1.3 (Compact Hausdorff Separation Principle). *Let $\mathscr{K}$ be a class of Hausdorff structures and let $\mathbf{Y}$ be a nontrivial compact structure. Then* $\mathbf{Y} \in \mathbf{IS}_c\mathbf{P}\mathscr{K}$ *if and only if $\mathbf{Y}$ satisfies* (i), (ii) *and* (iii) *of Lemma 1.2.*

Proof. By the proof of Lemma 1.2, it clearly suffices to show that under the assumption of (i), (ii) and (iii) of Lemma 1.2 ψ^{-1} is continuous. This however follows from the fact that Y is a compact space, ψ is an injective mapping and $\prod\langle X_i \mid i\in I\rangle$ is a Hausdorff space.

Corollary 1.4 (Compact Hausdorff Separation Principle for Algebras). *Let $\mathscr{K}$ be a class of Hausdorff algebras and let* $\mathbf{Y}$ *be a nontrivial compact algebra. Then* $\mathbf{Y}\in\mathbf{IS}_c\mathbf{P}\mathscr{K}$ *if and only if distinct members of Y can be separated by continuous homomorphisms into members of $\mathscr{K}$.*

Remark 1.5. DAVEY and WERNER [7] denote conditions (i) and (iii) of Lemma 1.2 by (SEP) and claim that a compact $\mathbf{Y}$ belongs to $\mathbf{IS}_c\mathbf{P}\mathscr{K}$ just in case it satisfies (SEP). In the presence of partial operations this is not correct and it requires a good deal of technical detail work to correct their arguments in this case. Unfortunately we shall have to get involved with this issue in Section 3.

Now we introduce just enough language to determine $\mathbf{IS}_c\mathbf{P}\mathscr{K}$ as a topological quasi variety. Let Vb be a *proper* class of *variables*. We define the class Tm of finitary *terms* as usual building up terms from variables using both operation and partial operation symbols. We have two types of *atomic formulas*. First the (finitary) *algebraic* and *relational* atomic formulas

$$\tau \approx \sigma, \quad r\tau_0\tau_1\ldots\tau_{n-1},$$

where $\tau, \sigma, \tau_0, \ldots, \tau_{n-1}\in Tm$, $\approx$ is the identity symbol and $r\in \mathbf{Rl}$; and secondly the (infinitary) *topological* atomic formulas

$$\tau_d \xrightarrow{d\in D} \sigma,$$

where $\langle D, \leqq\rangle$ is a directed set, $\tau\colon D\to Tm$ is a net in Tm and $\sigma\in Tm$. A *topological quasi atomic formula* is an expression of the form

$$\bigwedge\{\Phi_\xi \mid \xi\in\Delta\} \Rightarrow \Psi \quad \text{or} \quad \bigvee\{\neg\Phi_\xi \mid \xi\in\Delta\},$$

where $\{\Phi_\xi \mid \xi\in\Delta\}$ is a (possibly empty) *set* of atomic formulas and Ψ is an atomic formula.

An *assignment* of the variables in the topological structure $\mathbf{X}$ is a mapping $x\colon Vb\to X$. For $\tau\in Tm$ we define by simultaneous recursion the two notions "τ is defined for x (in X)" and "$\tau^{\mathbf{X}}[x]\in X$" in case τ is defined for x.

(1) If $v\in Vb$, then v is defined for x and $v^{\mathbf{X}}[x]=x(v)$.

(2) If $f\in\mathbf{Op}$, then $f\tau_0\ldots\tau_{n-1}$ is defined for x iff $\tau_0, \ldots, \tau_{n-1}$ are defined for x, and then $f\tau_0\ldots\tau_{n-1}{}^{\mathbf{X}}[x]=f^{\mathbf{X}}(\tau_0^{\mathbf{X}}[x], \ldots, \tau_{n-1}^{\mathbf{X}}[x])$.

(3) If $g\in\mathbf{POp}$, then $g\tau_0\ldots\tau_{n-1}$ is defined for x iff $\tau_0, \ldots, \tau_{n-1}$ are defined for x and $\langle\tau_0^{\mathbf{X}}[x], \ldots, \tau_{n-1}^{\mathbf{X}}[x]\rangle\in\operatorname{dom}(g^{\mathbf{X}})$, and then $g\tau_0\ldots\tau_{n-1}{}^{\mathbf{X}}[x]=g^{\mathbf{X}}(\tau_0^{\mathbf{X}}[x], \ldots, \tau_{n-1}^{\mathbf{X}}[x])$.

Next we define the notion of *satisfaction* for quasi atomic formulas.

$$\mathbf{X} \models \sigma \approx \tau[x]$$

if σ and τ are defined for x and $\sigma^{\mathbf{X}}[x]=\tau^{\mathbf{X}}[x]$.

$$\mathbf{X} \models r\tau_0 \dots \tau_{n-1}[x]$$

if $\tau_0, \dots, \tau_{n-1}$ are defined for x and $\langle \tau_0^{\mathbf{X}}[x], \dots, \tau_{n-1}^{\mathbf{X}}[x] \rangle \in r^{\mathbf{X}}$.

$$\mathbf{X} \models \tau_d \xrightarrow{d\in D} \sigma[x]$$

if all τ_d, $d \in D$, and σ are defined for x and $\tau_d^{\mathbf{X}}[x] \xrightarrow{d\in D} \sigma^{\mathbf{X}}[x]$.

$$\mathbf{X} \models \wedge\{\Phi_\xi \mid \xi \in \Delta\} \Rightarrow \Psi[x]$$

if there exists $\xi \in \Delta$ such that $\mathbf{X} \not\models \Phi_\xi[x]$ or $\mathbf{X} \models \Psi[x]$. Finally

$$\mathbf{X} \models \vee\{\neg\Phi_\xi \mid \xi \in \Delta\}[x]$$

provided $\mathbf{X} \not\models \Phi_\xi[x]$ for each $\xi \in \Delta$. Notice that for each "disjunctive" topological quasi atomic formula Φ there is a finite set of "implicational" topological quasi atomic formulas which are equivalent to Φ in any nontrivial structure.

A topological structure $\mathbf{X}$ is called a *model* of a class Σ of topological quasi atomic formulas (in symbols $\mathbf{X} \models \Sigma$) if for every $\Phi \in \Sigma$ and every $x\colon Vb \to X$, $\mathbf{X} \not\models \Phi[x]$. A class $\mathscr{K}$ of (compact) topological structures is called a *(compact) topological quasi variety* if there exists a class Σ of topological quasi atomic formulas such that $\mathscr{K}$ is the class of (compact) models of Σ. The *topological quasi atomical theory* of $\mathscr{K}$ (denoted by $\mathrm{Th}_{\mathrm{tqa}}\mathscr{K}$) is the class of topological quasi atomic formulas Φ such that each member of $\mathscr{K}$ is a model of Φ. The (compact) topological quasi variety *generated* by $\mathscr{K}$ is the class of (compact) models of $\mathrm{Th}_{\mathrm{tqa}}\mathscr{K}$.

Example 1.6. Some facts which hold in all topological structures by their very definition are expressible by topological quasi atomic formulas which therefore become *logically true*. As usual we shall write $\models \Phi$ in case $\mathbf{X} \models \Phi$ for all topological structures. To begin with, notice that for $g \in \mathbf{POp}$ and $x \in X^n$ we have

$$x \in \operatorname{dom}(g^{\mathbf{X}}) \quad \text{iff} \quad \mathbf{X} \models gv_0 \dots v_{n-1} \approx gv_0 \dots v_{n-1}[x]$$

so that $\not\models gv_0 \dots v_{n-1} \approx gv_0 \dots v_{n-1}$! On the other hand,

$$\models gv_0 \dots v_{n-1} \approx v_n \Rightarrow gv_0 \dots v_{n-1} \approx gv_0 \dots v_{n-1}$$

so that for $x \in X^{n+1}$ we have

$$x \in \operatorname{graph}(g^{\mathbf{X}}) \quad \text{iff} \quad \mathbf{X} \models gv_0 \dots v_{n-1} \approx v_n[x].$$

Now let $\langle D, \leqq\rangle$ be a directed set and $v\colon D \to Vb^n$. Then

$$\models [\bigwedge_{d\in D} g(v(d)_0, \ldots, v(d)_{n-1}) \approx g(v(d)_0, \ldots, v(d)_{n-1}) \wedge \bigwedge_{i<n} v(d)_i \xrightarrow{d\in D} u_i] \Rightarrow$$

$$\Rightarrow [g(v(d)_0, \ldots, v(d)_{n-1}) \xrightarrow{d\in D} g(u_0, \ldots, u_{n-1})]$$

because for each topological structure $\mathbf{X}$, $g^{\mathbf{X}}$ is continuous. Similarly we can express with topological quasi atomic formulas that $f^{\mathbf{X}}$ is continuous and $r^{\mathbf{X}}$ is closed, where $f\in\mathbf{Op}$ and $r\in\mathbf{Rl}$. We can also say "the graph of $g^{\mathbf{X}}$ is closed", where $g\in\mathbf{POp}$:

$$[\bigwedge_{d\in D} g(v(d)_0, \ldots, v(d)_{n-1}) \approx v(d)_n \wedge \bigwedge_{i\leqq n} v(d)_i \xrightarrow{d\in D} u_i] \Rightarrow g(u_0, \ldots, u_{n-1}) \approx u_n.$$

Let this formula be abbreviated by $\mathrm{Cl}(g)$. Now $\not\models \mathrm{Cl}(g)$, but $\mathbf{X}\models \mathrm{Cl}(g)$ in case $\mathbf{X}$ is Hausdorff and $g^{\mathbf{X}}$ is a full operation on X. Of course, this observation applies to the graphs of operations as well.

Example 1.7. Consider the class of topological abelian groups $\langle A, +, -, 0\rangle$. Then a discrete abelian group is *torsion* if and only if it is a model of the single topological atomic formula $m!v \xrightarrow{m\in\omega} 0$.

Next we shall consider *preservation properties.*

Lemma 1.8. (i) *Suppose* $\mathbf{X}\subseteq\mathbf{Y}$, τ *is a term and* $x\colon Vb\to X$. *Then* τ *is defined for* x *in* $\mathbf{X}$ *if and only if* τ *is defined for* x *in* $\mathbf{Y}$. *Moreover, in this case* $\tau^{\mathbf{X}}[x] = \tau^{\mathbf{Y}}[x]$.

(ii) *Suppose* $\varphi\colon \mathbf{X}\to\mathbf{Y}$, τ *is a term and* $x\colon Vb\to X$. *If* τ *is defined for* x, *then* τ *is defined for* $\varphi\circ x$ *and* $\varphi\tau^{\mathbf{X}}[x]=\tau^{\mathbf{Y}}[\varphi\circ x]$.

(iii) *Suppose* $\mathbf{Y}=\prod\langle \mathbf{X}_i \mid i\in I\rangle$, τ *is a term and* $x\colon Vb\to Y$. *Then* τ *is defined for* x *if and only if for every* $i\in I$, τ *is defined for* $\eta_i\circ x$. *Moreover, in this case* $\tau^{\mathbf{Y}}[x](i)=\tau^{\mathbf{X}_i}[\pi_i\circ x]$ *for all* $i\in I$.

Lemma 1.9. (i) *Suppose* $\mathbf{X}\subseteq\mathbf{Y}$, Φ *is an atomic formula and* $x\colon Vb\to X$. *Then* $\mathbf{X}\models\Phi[x]$ *iff* $\mathbf{Y}\models\Phi[x]$.

(ii) *Suppose* $\varphi\colon \mathbf{X}\to\mathbf{Y}$, Φ *is an atomic formula and* $x\colon Vb\to X$. *Then* $\mathbf{X}\models\Phi[x]$ *implies* $\mathbf{Y}\models\Phi[\varphi\circ x]$.

(iii) *Suppose* $\mathbf{Y}=\prod\langle \mathbf{X}_i \mid i\in I\rangle$, Φ *is an atomic formula and* $x\colon Vb\to Y$. *Then* $\mathbf{Y}\models\Phi[x]$ *iff* $\mathbf{X}_i\models\Phi[\pi_i\circ x]$ *for all* $i\in I$.

Corollary 1.10. *Topological quasi atomic formulas are preserved by* **I**, **S** *and* **P**.

Now we prove a technical lemma which characterizes continuous homomorphisms with a fixed domain $\mathbf{Y}$, where we have $Y\subseteq Vb$. Choose $a\colon Vb\to Y$ where

$a(v)=v$ for $v\in Y$. Next define Σ to be the *set* of all algebraic and relational atomic formulas Φ with variables from Y where $\mathbf{Y}\models\Phi[a]$, and all topological atomic formulas $\tau_d \overset{d\in D}{\to} \sigma$ with variables from Y, where D is a subset of the power set of Y directed by inclusion and $\mathbf{Y}\models\tau_d \overset{d\in D}{\to} \sigma[a]$.

Lemma 1.11. *A map* $\varphi\colon Y\to X$ *is a continuous homomorphism if and only if for each* $\Phi\in\Sigma$, $\mathbf{X}\models\Phi[\varphi\circ a]$.

Proof. One direction follows from Lemma 1.9 (ii), so assume $\varphi\colon Y\to X$ and $\mathbf{X}\models\Phi[\varphi\circ a]$ for each $\Phi\in\Sigma$. We claim that $\varphi\colon \mathbf{Y}\to\mathbf{X}$. Indeed, consider $g\in\mathbf{POp}$ and $\langle x_0, \dots, x_{n-1}\rangle\in\operatorname{dom}(g^{\mathbf{Y}})$. Then

$$\mathbf{Y}\models gx_0\dots x_{n-1}\approx gx_0\dots x_{n-1}[a]$$

and therefore

$$\mathbf{X}\models gx_0\dots x_{n-1}\approx gx_0\dots x_{n-1}[\varphi\circ a].$$

It follows that $\langle\varphi x_0, \dots, \varphi x_{n-1}\rangle\in\operatorname{dom}(g^{\mathbf{X}})$. Considering the atomic formula $gx_0\dots x_{n-1}\approx x_n$, where $g^{\mathbf{Y}}(x_0, \dots, x_{n-1})=x_n$, we establish that

$$\varphi g^{\mathbf{Y}}(x_0, \dots, x_{n-1}) = g^{\mathbf{X}}(\varphi x_0, \dots, \varphi x_{n-1}).$$

Operations and relations are treated similarly so that we are left with showing that φ is continuous. So suppose $W\subseteq X$ is closed and $z\in Y$ is a limit point of $\varphi^{-1}(W)$. Let D be the neighborhood system of y. D is directed by inclusion. For each $U\in D$ there exists $\delta U\in U\cap\varphi^{-1}(W)$, and $\delta U \overset{U\in D}{\to} z$. In other words, $\mathbf{Y}\models\delta U \overset{U\in D}{\to} z[a]$, so that $\mathbf{X}\models\delta U \overset{U\in D}{\to} z[\varphi\circ a]$. Thus $\varphi\circ\delta U \overset{U\in D}{\to} \varphi z$, where $\varphi\circ\delta U\in W$ for all $U\in D$. Since W is closed, $\varphi z\in W$ and therefore $z\in\varphi^{-1}(W)$.

Theorem 1.12. $\mathbf{Y}\in\mathbf{ISP}\mathscr{K}$ *if and only if* $\mathbf{Y}\models\mathrm{Th}_{\mathrm{tqa}}\mathscr{K}$.

Proof. Assume $\mathbf{Y}\models\mathrm{Th}_{\mathrm{tqa}}\mathscr{K}$. We shall establish conditions (i)—(iv) of the Separation Principle. Without loss of generality we may assume that $Y\subseteq Vb$. Let $x, y\in Y$, where $x\neq y$, and define Σ as in Lemma 1.11. Then

$$\mathbf{Y}\not\models \bigwedge\{\Phi\mid\Phi\in\Sigma\}\Rightarrow x\approx y[a]$$

and therefore there exist $\mathbf{X}\in K$ and $b\colon Vb\to X$ such that

$$\mathbf{X}\not\models \bigwedge\{\Phi\mid\Phi\in\Sigma\}\Rightarrow x\approx y[b].$$

Define $\varphi\colon Y\to X$ by $\varphi(v)=b(v)$, $v\in Y$. By the choice of variables we may assume that $b=\varphi\circ a$. Since $\mathbf{X}\models\Phi[\varphi\circ a]$ for each $\Phi\in\Sigma$, $\varphi\colon \mathbf{Y}\to\mathbf{X}$ by Lemma 1.11. Since $\varphi(x)\neq\varphi(y)$, condition (i) of the Separation Principle is established. The other conditions are proven by the same argument. Now check that the assertion is also true when $\mathbf{Y}$ is trivial.

Corollary 1.13. *A class $\mathcal{K}$ of (compact) topological structures is a (compact) topological quasi variety if and only if $\mathcal{K}$ is closed under* $\mathbf{I}, \mathbf{S}_{(c)}$, *and* $\mathbf{P}$.

Example 1.14. Since a subspace of a compact topological space is not necessarily compact, by Theorem 1.8, the class of *compact* topological structures is *not* in general a topological quasi variety. This forced us to introduce the notion of a *compact topological quasi variety* in the metalanguage.

As mentioned in the introduction, MAL'CEV [17] shows that a class $\mathcal{K}$ of *algebras* is a quasi variety if and only if it is closed under the formation of isomorphic images, subalgebras and reduced products, and ANDRÉKA, BURMEISTER and NÉMETI [1] have generalized this to *partial algebras*. It turns out that there is no corresponding result for topological quasi varieties because they are not closed under ultrapowers.

Lemma 1.15. *Let* $\mathbf{X}$ *be any topological structure and let U be a nonprincipal ultrafilter on an infinite set I. Then the quotient topology on X_U^I is the indiscrete topology.*

Proof. Let $\varphi\colon X^I \to X_U^I$ be the canonical mapping, $M \subseteq X^I$ a basic open set in the product space, and let $x \in X^I$. Choose $y \in M$ which differs from x on a finite set. Since U is nonprincipal, $\{i \in I \mid x(i) = y(i)\} \in U$. Thus $x/U = y/U \in \varphi(M)$ so that $\varphi(M) = X_U^I$. The assertion follows at once.

Now a definition of the ultraproduct $\mathbf{X}_U^I$ can only be considered adequate if the canonical homomorphism $\varphi\colon \mathbf{X}^I \to \mathbf{X}_U^I$ is continuous. Thus the topology on X_U^I has to be trivial. Hence, if $\mathbf{X}$ is a non-trivial Hausdorff structure, then $\mathbf{X}_U^I$ is not Hausdorff. It follows from Example 3.2 that no topological quasi variety containing a non-trivial Hausdorff structure is closed under ultrapowers.

2. Compact (topological) quasi varieties equivalent to (algebraic) quasi varieties

In this section we shall investiate a method of generating compact topological quasi varieties which has been recently developed in duality theory. This method yields many interesting examples of compact quasi varieties which play an important role in the literature. The foundations of (topological) duality theory were laid in DAVEY [6]. In a recent paper DAVEY and WERNER [7] give an expansive exposition of duality theory which contains some important advances yielding new applications. The idea of central interest to us is their notion of *full duality*. Unfortunately DAVEY and WERNER [7] contains some claims concerning full duality whose proofs are not correct in case partial operations are involved. This then yields some applications that are not justified. In this section we shall develop a theory of full duality which hopefully is both correct and substantially contributes to better understanding of this notion. Moreover, our approach yields new full duality results which vindicate and generalize the unestablished claims of Davey and Werner. For this purpose we shall (almost) completely adopt the notation and terminology of Davey and Werner [7] and shall briefly review the setting of their investigation.

To begin with, we shall simultaneously deal with two distinct similarity types: A type t of *algebras* determined by a set Op of operation symbols, and a type $\mathbf{t}$ of topological structures determined by a set $\mathbf{Op}$ of operation symbols, a set $\mathbf{POp}$ of partial operation symbols and a set $\mathbf{Rl}$ of relation symbols. Given is a finite, non-empty set P, an algebra $\mathfrak{P}=\langle P, f^{\mathfrak{P}}\rangle_{f\in Op}$ of similarity type t, and a topological structure $\mathbf{P}=\langle P, f^{\mathbf{P}} g^{\mathbf{P}}, r^{\mathbf{P}}\rangle_{f\in \mathbf{Op},\, g\in \mathbf{POp},\, r\in \mathbf{Rl}}$ of similarity type $\mathbf{t}$, where P is endowed with the discrete topology. In addition we require that $\mathbf{P}$ and $\mathfrak{P}$ satisfy the two equivalent conditions of the next lemma:

Lemma 2.1. *The following are equivalent:*

(i) *Each operation, non-empty partial operation and non-empty relation of* $\mathbf{P}$ *determines a subalgebra of a power of* $\mathfrak{P}$.

(ii) *Each operation of* $\mathfrak{P}$ *is a continuous homomorphism from a power of* $\mathbf{P}$ *into* $\mathbf{P}$.

The purpose of this requirement is to secure Lemma 2.2.

We now consider the *(algebraic) quasi variety* $\mathscr{L}=\mathbf{ISP}\mathfrak{P}$ as a category, where for each $\mathfrak{A}, \mathfrak{B}\in\mathscr{L}$, $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ denotes the set of homomorphisms $f\colon \mathfrak{A}\to\mathfrak{B}$. Simultaneously we consider the *compact (topological) quasi variety* $\mathscr{R}_0=\mathbf{IS}_c\mathbf{P}\mathbf{P}$ as a category, where for each $\mathbf{X}, \mathbf{Y}\in\mathscr{R}_0$, $\mathscr{R}_0(\mathbf{X}, \mathbf{Y})$ denotes the set of continuous homomorphisms $\varphi\colon \mathbf{X}\to\mathbf{Y}$. Notice, since P is a compact Hausdorff space, $\mathbf{X}\in\mathbf{S}_c\mathbf{P}\mathbf{P}$ if and only if $\mathbf{X}$ is a closed substructure of a direct power of $\mathbf{P}$.

Lemma 2.2. (Davey and Werner [7]). (i) *For each* $\mathfrak{A}\in\mathscr{L}$, $\mathscr{L}(\mathfrak{A}, \mathfrak{P})$ *determines a compact substructure of* $\mathbf{P}^A$.

(ii) *For each* $\mathbf{X}\in\mathscr{R}_0$, $\mathscr{R}_0(\mathbf{X}, \mathbf{P})$ *determines a subalgebra of* $\mathfrak{P}^X$.

For each $\mathfrak{A}\in\mathscr{L}$, let $D(\mathfrak{A})$ be the compact substructure of $\mathbf{P}^A$ determined by $D(A)=\mathscr{L}(\mathfrak{A}, \mathfrak{P})$, and for each $\mathbf{X}\in\mathscr{R}_0$, let $E(\mathbf{X})$ be the subalgebra of $\mathfrak{P}^X$ determined by $E(X)=\mathscr{R}_0(\mathbf{X}, \mathbf{P})$. For each $f\in\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ and $g\in D(B)$ define $D(f)(g)= =g\circ f$, and for each $\varphi\in\mathscr{R}_0(\mathbf{X}, \mathbf{Y})$ and $\psi\in E(Y)$ define $E(\varphi)(\psi)=\psi\circ\varphi$.

Lemma 2.3. (Davey and Werner [7]). *D is a contravariant functor, i.e.*

(i) $D(\mathfrak{A})\in\mathscr{R}_0$,

(ii) $D(f)\colon D(\mathfrak{B}) \to D(\mathfrak{A})$,

(iii) $D(f\circ g) = D(g)\circ D(f)$.

Lemma 2.4 (Davey and Werner [7]). *E is a contravariant functor, i.e.*

(i) $E(\mathbf{X})\in\mathscr{L}$,

(ii) $E(\varphi)\colon E(\mathbf{Y}) \to E(\mathbf{X})$,

(iii) $E(\varphi\circ\psi) = E(\psi)\circ E(\varphi)$.

This setting suggests a natural concept of duality. E is called a *dual representation* for $\mathscr{L}$ if E is *onto objects,* i.e. for each $\mathfrak{A}\in\mathscr{L}$ there exists $\mathbf{X}\in\mathscr{R}_0$ such that $\mathfrak{A}\cong E(\mathbf{X})$. Davey and Werner [7] set up a situation where such a dual representation is achieved *canonically*. For each $\mathfrak{A}\in\mathscr{L}$ and $a\in A$ define the projection $e_{\mathfrak{A}}(a)=\pi_a\colon D(\mathfrak{A})\to\mathbf{P}$, and for each $\mathbf{X}\in\mathscr{R}_0$ and $x\in X$ define the projection $\varepsilon_{\mathbf{X}}(x)=$ $=\pi_x\colon E(\mathbf{X})\to\mathfrak{P}$.

Lemma 2.5 (Davey and Werner [7]). *D and E are adjoint to each other, and $e_{\mathfrak{A}}$ and $\varepsilon_{\mathbf{X}}$ are embeddings in $\mathscr{L}$ and $\mathscr{R}_0$ respectively, i.e.*

(i) $e_{\mathfrak{A}}\colon \mathfrak{A} \Vert\!\!\longrightarrow ED(\mathfrak{A})$.

(ii) $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \Vert\!\!\longrightarrow DE(\mathbf{X})$ *is a topological embedding onto a closed subspace of* $\mathbf{P}^{E(X)}$.

(iii) *For each $h\in\mathscr{L}(\mathfrak{A},\mathfrak{B})$ and $\varphi\in\mathscr{R}_0(\mathbf{X},\mathbf{Y})$, the following diagrams commute:*

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "h"] \arrow[d, "e_{\mathfrak{A}}"'] & \mathfrak{B} \arrow[d, "e_{\mathfrak{B}}"] \\ ED(\mathfrak{A}) \arrow[r, "ED(h)"'] & ED(\mathfrak{B}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{X} \arrow[r, "\varphi"] \arrow[d, "\varepsilon_{\mathbf{X}}"'] & \mathbf{Y} \arrow[d, "\varepsilon_{\mathbf{Y}}"] \\ DE(\mathbf{X}) \arrow[r, "DE(\varphi)"'] & DE(\mathbf{Y}) \end{tikzcd}$$

(iv) *There is a one-to-one correspondence between $\mathscr{L}(\mathfrak{A}, E(\mathbf{X}))$ and $\mathscr{R}_0(\mathbf{X}, D(\mathfrak{A}))$ defined by the commuting diagrams*

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "e_{\mathfrak{A}}"] \arrow[dr, "g"'] & ED(\mathfrak{A}) \arrow[d, "E(\varphi)"] \\ & E(\mathbf{X}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{X} \arrow[r, "\varepsilon_{\mathbf{X}}"] \arrow[dr, "\varphi"'] & DE(\mathbf{X}) \arrow[d, "D(g)"] \\ & D(\mathfrak{A}) \end{tikzcd}$$

i.e. $g=E\big(D(g)\circ\varepsilon_{\mathbf{X}}\big)\circ e_{\mathfrak{A}}$ and $\varphi=D\big(E(\varphi)\circ e_{\mathfrak{A}}\big)\circ\varepsilon_{\mathbf{X}}$.

In this setting Davey and Werner [7] define their notions of duality and full duality.

Definition 2.6 (Davey and Werner [7]). (D, E) is called a *duality* if for every $\mathfrak{A}\in\mathscr{L}$, $e_{\mathfrak{A}}\colon \mathfrak{A} \Vert\!\!\longrightarrow ED(\mathfrak{A})$ is an isomorphism.

Clearly, if (D, E) is a duality, then E is a dual representation for $\mathscr{L}$. Moreover, in this case for each $\mathfrak{A}\in\mathscr{L}$ there is a *canonical* choice for a representative in $\mathscr{R}_0$, namely $D(\mathfrak{A})$. Thus all members of the (algebraic) quasi variety $\mathbf{ISP}\mathfrak{P}$ are *uniformly* represented as algebras of continuous functions.

The notion of "full duality" concerns the "uniqueness" of the representation. For this purpose Davey and Werner [7] consider full subcategories $D(\mathscr{L})\subseteq\mathscr{S}\subseteq\mathscr{R}_0$.

Definition 2.7 (DAVEY and WERNER [7]). Let (D, E) be a duality and let $D(\mathscr{L})\subseteq\mathscr{S}\subseteq\mathscr{R}_0$ be a full subcategory. (D, E) is called *full* between $\mathscr{L}$ and $\mathscr{S}$ if for every $\mathbf{X}\in\mathscr{S}$, $\varepsilon_{\mathbf{X}}$: $\mathbf{X} \Vdash\!\!\twoheadrightarrow DE(\mathbf{X})$ is a topological isomorphism.

Now it appears to us that this relativized notion of full duality is rather misleading because in a sense it is completely superfluous. This situation then tends to distract from the real "issue of full duality". To say more precisely what we mean, let $D(\mathscr{L})\subseteq\mathscr{S}\subseteq\mathscr{R}_0$ be a full subcategory. $E_{\mathscr{S}}$ denotes the restriction of E to $\mathscr{S}$, and is called a *category anti-equivalence* between $\mathscr{S}$ and $\mathscr{L}$ if it is onto objects, *full* (i.e. for any $\mathbf{X}, \mathbf{Y}\in\mathscr{S}$ and any h: $E(\mathbf{Y})\to E(\mathbf{X})$, there exists φ: $\mathbf{X}\to\mathbf{Y}$ such that $h=E(\varphi)$) and *faithful* (i.e. for any $\varphi, \psi\in\mathscr{S}(\mathbf{X}, \mathbf{Y})$, if $E(\varphi)=E(\psi)$ then $\varphi=\psi$). In this setting the last condition is actually redundant:

Lemma 2.8. *E is faithful.*

Proof. Suppose $E(\varphi)=E(\psi)$, where φ, ψ: $\mathbf{X}\to\mathbf{Y}$, and let χ: $\mathbf{Y} \Vdash\!\!\twoheadrightarrow \mathbf{P}^I$. Then for each $i\in I$, $\pi_i\circ\chi\in E(Y)$. Thus for all $x\in X$,

$$E(\varphi)(\pi_i\circ\chi)(x) = E(\psi)(\pi_i\circ\chi)(x),$$
$$\pi_i\chi\varphi(x) = \pi_i\chi\psi(x),$$
$$\chi\varphi(x) = \chi\psi(x),$$
$$\varphi(x) = \psi(x).$$

This establishes that $\varphi=\psi$.

Similarly we say that D is a category anti-equivalence between $\mathscr{L}$ and $\mathscr{S}$ if it is onto objects, full and faithful. Again we may forget about the last condition.

Lemma 2.9. *D is faithful.*

Proof. Similar to the proof of Lemma 2.8.

Lemma 2.10. *Suppose (D, E) is a duality and $D(\mathscr{L})\subseteq\mathscr{S}\subseteq\mathscr{R}_0$ is a full subcategory. If (D, E) is full between $\mathscr{L}$ and $\mathscr{S}$, then D: $\mathscr{L}\to\mathscr{S}$ and $E_{\mathscr{S}}$: $\mathscr{S}\to\mathscr{L}$ are both category anti-equivalences which are inverse to each other in the sense that*

$$ED(\mathfrak{A}) \cong \mathfrak{A} \quad \textit{for all} \quad \mathfrak{A}\in\mathscr{L}; \quad DE(\mathbf{X}) \cong \mathbf{X} \quad \textit{for all} \quad \mathbf{X}\in\mathscr{S}.$$

Proof. By Definitions 2.6, 2.7 and Lemmas 2.8, 2.9 both D and E are onto objects and faithful. To show that D is full, let $\mathfrak{A}, \mathfrak{B}\in\mathscr{L}$ and φ: $D(\mathfrak{B})\to D(\mathfrak{A})$. Define $h=e_{\mathfrak{B}}^{-1}\circ E(\varphi)\circ e_{\mathfrak{A}}$. Then h: $\mathfrak{A}\to\mathfrak{B}$ and by Lemma 2.5 (iii),

$$E(\varphi) = e_{\mathfrak{B}}\circ h\circ e_{\mathfrak{A}}^{-1} = ED(h).$$

It follows from Lemma 2.8 that $\varphi=D(h)$. This establishes that D is full, and to show that E is full we argue similarly using Lemma 2.9.

Now if E is a dual representation for $\mathscr{L}$ then the "natural notion of uniqueness" is category anti-equivalence. Again in the Davey—Werner setting more is required: Each $\mathfrak{A} \in \mathscr{L}$ can be *canonically* recaptured from its canonical representation $D(\mathfrak{A})$. This is an extremely tight connection so that the next (purely category theoretical) observation is not surprising.

Lemma 2.11. *Suppose (D, E) is a duality and $D(\mathscr{L}) \subseteq \mathscr{S} \subseteq \mathscr{R}_0$ is a full subcategory. Then (D, E) is full between $\mathscr{L}$ and $\mathscr{S}$ if and only if $\mathbf{I}\mathscr{S} = \mathbf{I}D(\mathscr{L})$.*

Proof. If (D, E) is full between $\mathscr{L}$ and $\mathscr{S}$, then $\mathbf{I}\mathscr{S} = \mathbf{I}D(\mathscr{L})$ by definition. Conversely, consider $A \in \mathscr{L}$. In Lemma 2.5 (iv) set $\mathbf{X} = D(\mathfrak{A})$ and $\varphi = \mathrm{id}_{D(A)}$. Then we obtain the following commuting diagrams:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "e_{\mathfrak{A}}"] \arrow[dr, "g"'] & ED(\mathfrak{A}) \arrow[d, "E(\mathrm{id}_{D(A)})"] \\ & ED(\mathfrak{A}) \end{tikzcd} \qquad \begin{tikzcd} D(\mathfrak{A}) \arrow[r, "\varepsilon_{D(\mathfrak{A})}"] \arrow[dr, "\mathrm{id}_{d(A)}"'] & DED(\mathfrak{A}) \arrow[d, "D(g)"] \\ & D(\mathfrak{A}) \end{tikzcd}$$

By Lemma 2.4, $E(\mathrm{id}_{D(A)}) = \mathrm{id}_{ED(A)}$ and therefore $g = \mathrm{id}_{ED(A)} \circ e_{\mathfrak{A}} = e_{\mathfrak{A}}$. It follows that $D(g)$ is bijective and hence $\varepsilon_{D(\mathfrak{A})}$ is bijective. Now we obtain at once from Lemma 2.5 (iii) that for any $\mathbf{X} \in \mathbf{I}D(\mathscr{L})$, $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \Vdash\!\!\rightarrow DE(\mathbf{X})$ is a topological isomorphism.

This observation reveals that any duality is full in exactly one way. The "issue of full duality" appears to be the task of *identifying* the category $\mathbf{I}D(\mathscr{L})$, i.e., to give a *comprehensible description* of the topological structures belonging to this category *in terms of their topology and their structure.* It does not appear that the Davey—Werner definition of full duality is helpful in this respect. In an attempt to carry out this task we shall first give a description of the category $\mathbf{I}D(\mathscr{L})$ which is completely independent from the category theoretical construction, i.e. which does not involve the adjoint contravariant functors D and E. For this purpose we introduce the notion of hull-kernel closed subset of a power of P. This notion plays an important role in sheaf representation and has been investigated in a much more general context in Krauss and Clark [16]. However, for our purposes it suffices to consider a limited version which we shall give a self contained treatment. For each non-empty set S, let $\mathfrak{F}_S$ be the subalgebra of $\mathfrak{P}^{(P^S)}$ ($\mathbf{ISP}\mathfrak{P}$-freely) generated by the set $\{\pi_s \mid s \in S\}$ of projections. For $\sigma, \tau \in F_S$ define

$$\mathrm{Eq}(\sigma, \tau) = \{x \in P^S \mid \sigma(x) = \tau(x)\}$$

and for an arbitrary subset $X \subseteq P^S$ define the *hull-kernel closure* $\bar{X}$ of X by

$$\bar{X} = \bigcap \{\mathrm{Eq}(\sigma, \tau) \mid X \subseteq \mathrm{Eq}(\sigma, \tau)\}.$$

In other words, $y \in \bar{X}$ if and only if $\sigma(y)=\tau(y)$ whenever $\sigma, \tau \in F_S$ and $\sigma(x)=\tau(x)$ for all $x \in X$. In particular, each member of $\pi_X F_S$ has a unique extension in $\pi_{\bar{X}} F_S$ so that $\pi_X \mathfrak{F}_X \cong \pi_{\bar{X}} \mathfrak{F}_S$. X is called *hull-kernel closed* if $X=\bar{X}$.

Lemma 2.12. *For any $X, Y \subseteq P^S$,*
(i) $X \subseteq \bar{X}$,
(ii) $\bar{\bar{X}}=\bar{X}$,
(iii) $X \subseteq Y$ *implies* $\bar{X} \subseteq \bar{Y}$.

Lemma 2.13. *Every hull-kernel closed subset of P^S is closed in the product topology.*

First we shall give a characterization of hull-kernel closed sets which will reveal the role of partial operations in the topological structures of similarity type $\mathbf{t}$.

Lemma 2.14. *Suppose $S \neq \emptyset$ and $X \subseteq P^S$. If for every $f\colon \pi_X \mathfrak{F}_S \to \mathfrak{P}$ there exists $x \in X$ such that $f=\pi_x$, then X is hull-kernel closed.*

Proof. If $y \in \bar{X}$, we can define $f\colon \pi_X \mathfrak{F}_S \to \mathfrak{P}$ by $f(\pi_X \sigma)=\sigma(y)$. Choose $x \in X$ so that $f=\pi_x$. Then for any $s \in S$,

$$x(s) = \pi_s(x) = \pi_x(\pi_s) = f(\pi_X(\pi_s)) = \pi_s(y) = y(s)$$

so that $y=x \in X$.

Now suppose $\mathfrak{A} \subseteq \mathfrak{P}^I$ and $f\colon \mathfrak{A} \to \mathfrak{P}$. We can view f as an I-place partial operation on P. For each non-empty S we can canonically lift f to an I-place partial operation $\bar{f}$ on P^S. The domain of $\bar{f}$ is defined by

$$\bar{A} = \{x \in (P^S)^I \mid \pi_s \circ x \in A \quad \text{for all} \quad s \in S\},$$

and $\bar{f}\colon \bar{A} \to P^S$ is defined by $\bar{f}(x)(s)=f(\pi_s \circ x)$. We call $X \subseteq P^S$ closed under $\bar{f}$ if $\bar{f}(x) \in X$ whenever $x \in X^I \cap \bar{A}$.

Lemma 2.15. *Suppose $S \neq \emptyset$ and $X \subseteq P^S$. Then X is hull-kernel closed if and only if X is closed under every $\bar{f}$, where $\mathfrak{A} \in \mathbf{SP}\mathfrak{P}$ and $f\colon \mathfrak{A} \to \mathfrak{P}$.*

Proof. Assume X is hull-kernel closed, $\mathfrak{A} \subseteq \mathfrak{P}^I$, $f\colon \mathfrak{A} \to \mathfrak{P}$ and $x \in X^I \cap \bar{A}$. Since $\mathfrak{F}_S$ is generated by $\{\pi_s \mid s \in S\}$, each member of F_S is of the form $\tau^{\mathfrak{F}_S}(\pi_{s_0}, \ldots, \pi_{s_{n-1}})$, where τ is an n-place term. So suppose

$$X \subseteq E(\tau^{\mathfrak{F}_S}(\pi_{s_0}, \ldots, \pi_{s_{n-1}}), \sigma^{\mathfrak{F}_S}(\pi_{s_0}, \ldots, \pi_{s_{n-1}})).$$

Then

$$\tau^{\mathfrak{A}}(\pi_{s_0} \circ x, \ldots, \pi_{s_{n-1}} \circ x)\colon I \to P$$

and therefore for each $i\in I$,

$$\tau^{\mathfrak{A}}(\pi_{s_0}\circ x, \dots, \pi_{s_{n-1}}\circ x)(i) = \tau^{\mathfrak{A}}(\pi_{s_0}x(i), \dots, \pi_{s_{n-1}}x(i)) =$$
$$= \tau^{\mathfrak{P}}(x(i)(s_0), \dots, x(i)(s_{n-1})) = \tau^{\mathfrak{F}_S}(\pi_{s_0}, \dots, \pi_{s_{n-1}})(x(i)) =$$
$$= \sigma^{\mathfrak{F}_S}(\pi_{s_0}, \dots, \pi_{s_{n-1}})(x(i)) = \sigma^{\mathfrak{A}}(\pi_{s_0}\circ x, \dots, \pi_{s_{n-1}}\circ x)(i).$$

Thus

$$\tau^{\mathfrak{F}_S}(\pi_{s_0}, \dots, \pi_{s_{n-1}})(\bar{f}(x)) = \tau^{\mathfrak{P}}(\bar{f}(x)(s_0), \dots, \bar{f}(x)(s_{n-1})) =$$
$$= \tau^{\mathfrak{P}}(f(\pi_{s_0}\circ x), \dots, f(\pi_{s_{n-1}}\circ x)) = f(\tau^{\mathfrak{A}}(\pi_{s_0}\circ x, \dots, \pi_{s_{n-1}}\circ x)) =$$
$$= f(\sigma^{\mathfrak{A}}(\pi_{s_0}\circ x, \dots, \pi_{s_{n-1}}\circ x)) = \sigma^{\mathfrak{F}_S}(\pi_{s_0}, \dots, \pi_{s_{n-1}})(\bar{f}(x)).$$

This establishes that $\bar{f}(x)\in\bar{X}=X$.

Conversely, suppose for all $\mathfrak{A}\in \mathbf{SP}\mathfrak{P}$ and $f\colon \mathfrak{A}\to\mathfrak{P}$, X is closed under $\bar{f}$. Consider $\pi_X\mathfrak{F}_S\subseteq\mathfrak{P}^X$ and $f\colon \pi_X\mathfrak{F}_S\to\mathfrak{P}$. Define $x\in P^S$ by $x(s)=f(\pi_X(\pi_s))$. We shall show that $x\in X$ and $f=\pi_x$. So consider $\overline{\pi_X F_S}\subseteq(P^S)^X$ and $\bar{f}\colon \overline{\pi_X F_S}\to P^S$. Let $\tau(y)=y$ for all $y\in X$. Clearly $\pi_s\circ\tau\in\pi_X F_S$ for all $s\in S$ and therefore $\tau\in\overline{\pi_X F_S}$. By hypothesis, $\bar{f}(\tau)\in X$ and for each $s\in S$,

$$\bar{f}(\tau)(s) = f(\pi_s\circ\tau) = f(\pi_X(\pi_s)) = x(s).$$

Thus $x=\bar{f}(\tau)\in X$. Finally, for each $s\in S$,

$$f(\pi_X(\pi_s)) = x(s) = \pi_s(x) = \pi_x(\pi_X(\pi_s)).$$

Since $\mathfrak{F}_S$ is generated by $\{\pi_s \mid s\in S\}$, $f=\pi_x$. By Lemma 2.14, X is hull-kernel closed.

Corollary 2.16. *Every hull-kernel closed set $X\subseteq P^S$, where $S\neq\emptyset$, determines a substructure $\mathbf{X}\subseteq\mathbf{P}^S$.*

Proof. Suppose $g\in\mathbf{POp}$. By the requirement Lemma 2.1 (i), $g^{\mathbf{P}}\subseteq P^{n+1}$ determines a subalgebra of $\mathfrak{F}^{n+1}$. Thus $D=\operatorname{dom}(g^{\mathbf{P}})$ determines a subalgebra $\mathfrak{D}\subseteq\mathfrak{P}^n$ and $g^{\mathbf{P}}\colon \mathfrak{D}\to\mathfrak{P}$. Thus by Lemma 2.15, X is closed under $g^{\mathbf{P}^S}\colon \bar{D}\to P^S$.

Lemma 2.17. *For every $\mathfrak{A}\in\mathscr{L}$, $D(A)\subseteq P^A$ is hull-kernel closed.*

Proof. Consider $\mathfrak{F}_A\subseteq\mathfrak{P}^{(P^A)}$. To simplify notation, we take $\mathfrak{P}=\langle P, +\rangle$, where $+$ is binary. Then for any $a, b\in A$, $\pi_a, \pi_b, \pi_{a+b}\in F_A$. Thus for any $f\in D(A)$,

$$\pi_{a+b}(f) = f(a+b) = f(a)+f(b) = \pi_a(f)+\pi_b(f) = (\pi_a+\pi_b)(f).$$

This shows that $D(A)\subseteq\operatorname{Eq}(\pi_{a+b}, \pi_a+\pi_b)$. It follows that for any $g\in\overline{D(A)}$, $\pi_{a+b}(g)=(\pi_a+\pi_b)(g)$ and therefore $g(a+b)=g(a)+g(b)$. Thus $g\in D(A)$ and $\overline{D(A)}\subseteq D(A)$.

Let $\mathbf{S}_{hk}\mathbf{PP}$ be the class of topological structures $\mathbf{X}$, where for some $S\neq\emptyset$, $X\subseteq P^S$ is hull-kernel closed, and let $\mathcal{R}_{hk}=\mathbf{IS}_{hk}\mathbf{PP}$.

Theorem 2.18. *$D(\mathcal{L})\subseteq\mathcal{R}_{hk}\subseteq\mathcal{R}_0$ is a full subcategory.*

Proof. Use Lemma 2.13, Corollary 2.16 and Lemma 2.17.

Next we shall characterize duality in terms of hull-kernel closed sets.

Lemma 2.19. *For each $f\in D(F_S)$ there exists $x\in P^S$ such that $f=\pi_x$.*

Proof. Define $x\in P^S$ by $x(s)=f(\pi_s)$. Then for any $s\in S$,

$$f(\pi_s) = x(s) = \pi_s(x) = \pi_x(\pi_s).$$

Since $\{\pi_s \mid s\in S\}$ generates $\mathfrak{F}_S$, $f=\pi_x$.

Lemma 2.20. *$\mathfrak{A}\in\mathbf{ISP}\mathfrak{P}$ if and only if there exist non-empty S and hull-kernel closed $X\subseteq P^S$ such that $\mathfrak{A}\cong\pi_X\mathfrak{F}_S$.*

Proof. Suppose $\mathfrak{A}\in\mathbf{ISP}\mathfrak{P}$. Then there exist non-empty S and $f\colon \mathfrak{F}_S\twoheadrightarrow\mathfrak{A}$, and there exist I and $g\colon \mathfrak{A}\,\|\!\!\rightarrow\mathfrak{P}^I$. For each $i\in I$ consider

$$\mathfrak{F}_S \xrightarrow{f} \mathfrak{A} \,\|\!\!\xrightarrow{g} \mathfrak{P}^I \xrightarrow{\pi_i} \mathfrak{P}.$$

Let $h_i\rightarrow\pi_i\circ g\circ f$. Then $h_i\in D(F_S)$ and by Lemma 2.19, there exist $x_i\in P^S$ such that $h_i=\pi_{x_i}$. Let $X=\{x_i \mid i\in I\}$. Then for any $\sigma,\tau\in F_S$ the following assertions are equivalent:

$$f(\sigma) = f(\tau).$$

$$\text{For all } i\in I,\ \pi_i gf(\tau)=\pi_i gf(\tau).$$

$$\text{For all } i\in I,\ h_i(\sigma)=h_i(\tau).$$

$$\text{For all } i\in I,\ \pi_{x_i}(\sigma)=\pi_{x_i}(\tau).$$

$$\pi_X(\sigma)=\pi_X(\tau).$$

Thus $\mathfrak{A}\cong\pi_X\mathfrak{F}_S\cong\pi_{\overline{X}}\mathfrak{F}_S$, and we may assume that $X=\overline{X}$.

Lemma 2.21. *For any non-empty S and closed $\mathbf{X}\subseteq\mathbf{P}^S$, $\pi_X\mathfrak{F}_S\subseteq E(\mathbf{X})$.*

Proof. Check directly that for each $s\in S$, $\pi_X(\pi_s)\colon X\to P$ is continuous. Since $\mathfrak{F}_S$ is generated by $\{\pi_s \mid s\in S\}$, by induction for each $\tau\in F_S$, $\pi_X(\tau)\colon X\to P$ is continuous. The remainder of the assertion follows from the requirement Lemma 2.1(iii).

Now in Lemma 2.5(iv) set $\mathfrak{A}=\pi_X\mathfrak{F}_S$ and $i\colon \pi_X\mathfrak{F}_S \Vdash\!\!\rightarrow E(\mathbf{X})$ the injection mapping to obtain the following commuting diagrams:

$$
\begin{tikzcd}
\pi_X\mathfrak{F}_S \arrow[r, "e_{\pi_X\mathfrak{F}_S}"] \arrow[dr, "i"'] & ED(\pi_X\mathfrak{F}_S) \arrow[d, "E(\delta_{\mathbf{X}})"] \\
& E(\mathbf{X})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{X} \arrow[r, "\varepsilon_{\mathbf{X}}"] \arrow[dr, "\delta_{\mathbf{X}}"'] & DE(\mathbf{X}) \arrow[d, "D(i)"] \\
& D(\pi_X\mathfrak{F}_S)
\end{tikzcd}
$$

Clearly $\delta_{\mathbf{X}}(x)=\pi_x\colon \pi_X\mathfrak{F}_S\to\mathfrak{P}$.

Lemma 2.22. $\delta_{\mathbf{X}}\colon \mathbf{X}\Vdash\!\!\rightarrow D(\pi_X\mathfrak{F}_S)$ *is a topological embedding onto a closed subspace of* $P^{\pi_X F_S}$.

Proof. Adjust the proof of Lemma 2.5 (ii).

Lemma 2.23. $\delta_{\mathbf{X}}\colon \mathbf{X}\Vdash\!\!\rightarrow D(\pi_X\mathfrak{F}_S)$ *is a topological isomorphism if and only if* X *is hull-kernel closed.*

Proof. If $\delta_{\mathbf{X}}$ is a surjective mapping then, by Lemma 2.14, X is hull-kernel closed. Conversely, assume X is hull-kernel closed. By Lemma 2.22 it suffices to show that $\delta_{\mathbf{X}}$ is a surjective mapping. Indeed, let $f\in D(\pi_X F_S)$ and consider

$$\mathfrak{F}_S \xrightarrow{\pi_X} \pi_X\mathfrak{F}_S \xrightarrow{f} \mathfrak{P}.$$

By Lemma 2.19, there exists $x\in P^S$ such that $f\circ\pi_X=\pi_x$. We shall show that $x\in X$. Consider $\sigma,\tau\in F_S$, where $X\subseteq \mathrm{Eq}\,(\sigma,\tau)$. Then $\pi_X(\sigma)=\pi_X(\tau)$ and therefore $f(\pi_X(\sigma))=f(\pi_X(\tau))$. Thus $\pi_x\sigma=\pi_x\tau$, i.e., $\sigma(x)=\tau(x)$. This establishes that $x\in\mathrm{Eq}\,(\sigma,\tau)$, so $x\in\overline{X}=X$. Now we obtain for any $s\in S$,

$$f(\pi_X(\pi_s)) = \pi_x(\pi_s) = \pi_x(\pi_X(\pi_s)).$$

Since $\{\pi_s \mid s\in S\}$ generates $\mathfrak{F}_S$, $f=\pi_x=\delta_{\mathbf{X}}(x)$.

Corollary 2.24. *If* X *is hull-kernel closed, then* $E(\mathbf{X})=\pi_X\mathfrak{F}_S$ *if and only if* $e_{\pi_X\mathfrak{F}_S}\colon \pi_X\mathfrak{F}_S\Vdash\!\!\rightarrow ED(\pi_X\mathfrak{F}_S)$ *is an isomorphism.*

Proof. Returning to the definition of $\delta_{\mathbf{X}}$ we obtain from Lemma 2.23 the following commuting diagrams:

$$
\begin{tikzcd}
\pi_X\mathfrak{F}_S \arrow[r, "e_{\pi_X\mathfrak{F}_S}"] \arrow[dr, "i"'] & ED(\pi_X\mathfrak{F}_S) \arrow[d, "E(\delta_{\mathbf{X}})", "="'] \\
& E(\mathbf{X})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{X} \arrow[r, "\varepsilon_{\mathbf{X}}"] \arrow[dr, "\delta_{\mathbf{X}}"'] & DE(\mathbf{X}) \arrow[d, "D(i)"] \\
& D(\pi_X\mathfrak{F}_S)
\end{tikzcd}
$$

The assertion follows at once.

Now we obtain the promised characterization of duality in terms of hull-kernel closed sets:

Theorem 2.25. *(D, E) is a duality if and only if for every non-empty S and every hull-kernel closed $X \subseteqq P^S$, $E(\mathbf{X}) = \pi_X \mathfrak{F}_S$.*

Proof. One direction follows immediately from Corollary 2.24, and for the other use Lemmas 2.5, 2.20 and Corollary 2.24.

Notice that in case (D, E) is a duality, then in particular for any non-empty S, $E(\mathbf{P}^S) = \mathfrak{F}_S$. Next we give a description of the dual category $\mathbf{I}D(\mathscr{L})$ which does not depend on the category theoretical construction:

Theorem 2.26. *Suppose (D, E) is a duality and $D(\mathscr{L}) \subseteqq \mathscr{S} \subseteqq \mathscr{R}_0$ is a full subcategory. Then (D, E) is full between $\mathscr{L}$ and $\mathscr{S}$ if and only if $\mathscr{S} = \mathscr{R}_{hk}$.*

Proof. By Theorem 2.25, for every non-empty S and every hull-kernel closed $X \subseteqq P^S$, $\varepsilon_{\mathbf{X}} = \delta_{\mathbf{X}}$. Thus by Lemma 2.24, (D, E) is full between $\mathscr{L}$ and $\mathscr{R}_{hk}$. The remainder of the assertion follows from Lemma 2.11.

Now the "dilemma of full duality" becomes apparent: The *definition* of hull-kernel closed structures $\mathbf{X} \in \mathbf{S}_{hk}\mathbf{P}\mathbf{P}$ does *not* involve the topology of the space X and the structure of $\mathbf{X}$ (in terms of the similarity type $\mathfrak{t}$!) but involves the $\mathbf{ISP}\mathfrak{P}$-free structures of similarity type t ! The code to translate between the two similarity types is given by Lemma 2.1. Unfortunately this code is so involved that the description "topological isomorphs of hull-kernel closed substructures of powers of $\mathbf{P}$" considered as a description of the category $\mathscr{R}_{hk}$ *in terms of the topology and the structure of its members* is so circuitous that it is practically incomprehensible. Now the authors have not been able to find a single example of a duality result in this setting where a comprehensible description of the dual category $\mathscr{R}_{hk}$ is given *in terms of the topology and the structure of its members* unless $\mathscr{R}_{hk} = \mathscr{R}_0$. Thus the "issue of full duality" appears to be the question:

"When is $\mathscr{R}_{hk} = \mathscr{R}_0$?"

which by Theorem 2.26 translates into the question:

"When is the duality full between $\mathscr{L}$ and $\mathscr{R}_0$?"

In fact every "full duality result" the authors are aware of implicitly involves showing $\mathscr{R}_{hk} = \mathscr{R}_0$. To make this explicit we shall conclude this section by describing those circumstances of duality which yield $\mathscr{R}_{hk} = \mathscr{R}_0$. Moreover, we shall see that under these circumstances we will always obtain the stronger conclusion $\mathbf{S}_{hk}\mathbf{P}\mathbf{P} = \mathbf{S}_c\mathbf{P}\mathbf{P}$, i.e. *closed subspaces of powers of $\mathbf{P}$ are hull-kernel closed.* Since all of this is relevant only in case (D, E) is a duality, we shall state an important result of DAVEY and WERNER [7] which yields duality in all cases considered by them. The key is their *interpolation condition:*

(IC) For all non-empty finite T and $\mathbf{X} \subseteq \mathbf{P}^T$, every ψ: $\mathbf{X} \to \mathbf{P}$ is the restriction of a T-ary term function of $\mathfrak{P}$ (i.e. $\pi_X \mathfrak{F}_T = E(\mathbf{X})$).

Theorem 2.27 (DAVEY and WERNER [7]). *If* $\mathbf{Op} \cup \mathbf{Rl}$ *is finite and* (IC) *holds, then* (D, E) *is a duality.*

Most applications of this theorem can be obtained from a special case which was proven independently by the authors and R. McKenzie. A $(k+1)$-ary term τ is called a *near-unanimity term* for $\mathfrak{P}$ if for any $a, b \in P$,

$$\tau^P(a, b, \ldots, b) = \tau^P(b, a, b, \ldots, b) = \cdots = \tau^P(b, \ldots, b, a) = b.$$

This notion was introduced by BAKER and PIXLEY [2]. They prove that (IC) holds in case $\mathfrak{P}$ has a $(k+1)$-ary near-unanimity term and $\mathbf{P}$ is chosen to have *all* subalgebras of $\mathfrak{P}^k$ as relations.

Corollary 2.28 (Clark, Krauss and McKenzie). *Suppose* $\mathfrak{P}$ *has a* $(k+1)$*-ary near unanimity term and let* $\mathbf{P} = \langle P, r \rangle_{r \in \mathrm{S}\mathfrak{P}^k}$. *Then* (D, E) *is a duality.*

Now we shall investigate circumstances yielding $\mathscr{R}_{hk} = \mathscr{R}_0$ under the hypothesis of Theorem 2.27, covering all "full duality results" that have come to our attention. We start with a strengthening of Theorem 2.27:

Theorem 2.29. *If* $\mathbf{Op} \cup \mathbf{Rl}$ *is finite and* (IC) *holds, then for every non-empty* S *and every closed* $\mathbf{X} \subseteq \mathbf{P}^S$, $E(\mathbf{X}) = \pi_X \mathfrak{F}_S$.

Proof. It is easy to verify that this is what DAVEY and WERNER [7] actually prove, although they don't state it.

Corollary 2.30. *If* $\mathbf{Op} \cup \mathbf{Rl}$ *is finite and* (IC) *holds, then* $\mathscr{R}_{hk} = \mathscr{R}_0$ *if and only if* $\mathrm{S}_{hk}\mathrm{P}\mathbf{P} = \mathrm{S}_c\mathrm{P}\mathbf{P}$, *i.e. "closed sets are hull-kernel closed".*

Proof. Suppose $\mathscr{R}_{hk} = \mathscr{R}_0$ and $\mathbf{X} \in \mathrm{S}_c\mathrm{P}\mathbf{P}$. By Theorem 2.26, $\varepsilon_{\mathbf{X}}$: $\mathbf{X} \Vdash\!\!\to DE(\mathbf{X})$ and therefore, by Theorem 2.29, $\varepsilon_{\mathbf{X}}$: $\mathbf{X} \Vdash\!\!\to D(\pi_X \mathfrak{F}_S)$. Thus $\varepsilon_{\mathbf{X}} = \delta_{\mathbf{X}}$ and X is hull-kernel closed by Lemma 2.23.

Thus in the setting determined by the hypothesis of Theorem 2.27, a duality (D, E) is *full* between $\mathscr{L}$ and $\mathscr{R}_0$ if and only if "closed sets are hull-kernel closed." Now DAVEY and WERNER [7] give sufficient conditions for this to occur which we shall look at next. Actually we shall take a little detour and look back at Lemma 2.15. This tells us that taking *all* homomorphisms from subalgebras of *arbitrary* powers of $\mathfrak{P}$ into $\mathfrak{P}$ as (infinitary) partial operations of $\mathbf{P}$ would even yield $\mathrm{S}_{hk}\mathrm{P}\mathbf{P} = \mathrm{SP}\mathbf{P}$. Now to begin with this would force us to introduce *infinitary* partial operations and to consider a similarity type with a *proper* class of partial operation symbols, clearly going beyond the formal scope of our setting. Moreover, the hypothesis of Theorem 2.27 actually commits us to *finitely* many *(finitary!)* partial operation symbols! On the other hand, notice that once we have obtained a duality from Theorem 2.27, then adding finitary $\mathfrak{P}$-homomorphisms to $\mathbf{Op} \cup \mathbf{POp}$ *while keeping* $\mathbf{POp}$ *finite* preserves (IC) and does not affect $\mathrm{S}_{hk}\mathrm{P}\mathfrak{P}$ whereas in general it will "cut down" $\mathrm{S}_c\mathrm{P}\mathbf{P}$. If we finally succeed to obtain $\mathrm{S}_{hk}\mathrm{P}\mathbf{P} = \mathrm{S}_c\mathrm{P}\mathbf{P}$ achieving full duality this way, then in general we will have blown up the similarity type $\mathbf{t}$ to a size which is too unwieldy for applications, so that the issue of "reducing the similarity type by removing redundancies" arises. There may also be some practical advantages to "rearranging the similarity type" by treating certain relations as partial operations or operations. Frequently all of this can be accomplished without disturbing full duality. Let F_1, F_2 be sets of operations, G_1, G_2 sets of partial operations and R_1, R_2 sets of relations on P respectively, where $F_2 \subseteq F_1$, $G_2 \subseteq G_1$ and $R_1 \subseteq R_2$. Suppose that $\mathbf{P}_1 = \langle P, f, g, r\rangle_{f\in F_1, g\in G_1, r\in R_1}$ and $\mathbf{P}_2 = \langle P, f, g, r\rangle_{f\in F_2, g\in G_2, r\in R_2}$ both satisfy the conditions of Lemma 2.1. We say that $F_1 \cup G_1 \cup R_1$ *generates* R_2 if for every non-empty finite T and $\mathbf{X} \subseteq \mathbf{P}_1^T$, if $\psi\colon \mathbf{X} \to \mathbf{P}_1$ then ψ preserves all relations of R_2 (and hence $\psi\colon \mathbf{X} \to \mathbf{P}_2$).

Lemma 2.31. *Suppose* $R_1 \cup G_1 \cup R_1$ *generates* R_2.

(i) *If* $\mathbf{P}_2$ *satisfies* (IC), *then* $\mathbf{P}_1$ *satisfies* (IC).

(ii) *If* $\mathrm{S}_{hk}\mathrm{P}\mathbf{P}_2 = \mathrm{S}_c\mathrm{P}\mathbf{P}_2$, *then* $\mathrm{S}_{hk}\mathrm{P}\mathbf{P}_1 = \mathrm{S}_c\mathrm{P}\mathbf{P}_1$.

Altogether these observations suggest the following

Full Duality Strategy 2.32. *Step 1:* Choose $\mathbf{t}$ such that $\mathbf{POp} \cup \mathbf{Rl}$ is finite and (IC) holds.

Step 2: Increase $\mathbf{Op} \cup \mathbf{POp}$ keeping $\mathbf{POp}$ finite until $\mathrm{S}_{hk}\mathrm{P}\mathbf{P} = \mathrm{S}_c\mathrm{P}\mathbf{P}$.

Step 3: Decrease $\mathbf{Rl}$, possibly treating certain relations as partial operations or operations, until $\mathbf{Op} \cup \mathbf{POp} \cup \mathbf{Rl}$ is a minimal generating set of the set of relations chosen in Step 1.

All full duality results we have looked at can actually be obtained following the Full Duality Strategy (where in some cases one or more steps may be skipped) and we shall present selected samples later in this section. First we shall give several tests to check whether Step 2 of this strategy has been successfully completed.

Lemma 2.33. *If* $\mathbf{POp} = \emptyset$, *then the following three conditions are equivalent:*

(i) $\mathrm{S}_{hk}\mathrm{P}\mathbf{P} = \mathrm{S}_c\mathrm{P}\mathbf{P}$.

(ii) *Every substructure of a finite power of* $\mathbf{P}$ *is hull-kernel closed.*

(HK) *For every non-empty finite* T, $\mathbf{X} \subseteq \mathbf{P}^T$ *and* $y \in P^T - X$, *there are two* T*-ary term functions of* $\mathfrak{P}$ *which agree on* X *but not at* y.

Proof. (HK) is just a full statement of (ii), and (i) implies (ii) trivially. So assume (HK) and let $\mathbf{X} \subseteq \mathbf{P}^S$ be closed, $z \in P^S - X$. Then there is a basic clopen set

$$U_T = \{x \in P^S | \pi_T z = \pi_T x\},$$

where $T \subseteq S$ is finite, $z \in U_T$ and $X \cap U_T = \emptyset$. It follows that $y = \pi_T z \notin \pi_T X$. Now there exists an embedding h: $\mathfrak{F}_T \Vert\!\!\longrightarrow \mathfrak{F}_S$ which sends the projection π_t, considered as a generator of $\mathfrak{F}_T$, to the projection π_t considered as a generator of $\mathfrak{F}_S$. Since $\mathbf{POp} = \emptyset$, $\pi_T X$ determines a substructure of $\mathbf{P}^T$. By hypothesis there exist σ, $\tau \in F_T$ such that σ and τ agree on $\pi_T X$ but not at y. Thus $h(\sigma)$ and $h(\tau)$ agree on X but not at z. This shows that X is hull-kernel closed and (i) is established.

Corollary 2.34. *Suppose* $\mathbf{POp} = \emptyset$, $\mathbf{Rl}$ *is finite and* (IC) *holds. Then the duality* (D, E) *is full between* $\mathscr{L}$ *and* $\mathscr{R}_0$ *if and only if* (HK) *holds.*

Proof. Use Theorem 2.26, Corollary 2.30 and Lemma 2.33.

Davey and Werner [7] proceed somewhat differently introducing the condition.

$(\mathrm{E3})_F$ If T is finite and $\mathbf{X} \subseteq \mathbf{Y} \subseteq \mathbf{P}^T$, where $X \neq Y$, then there are distinct φ, ψ: $\mathbf{X} \to \mathbf{P}$ which agree on X.

Now it turns out that in the presence of (IC) the conditions $(\mathrm{E3})_F$ and (HK) are equivalent:

Lemma 2.35. $(\mathrm{E3})_F + (\mathrm{IC}) \Rightarrow (\mathrm{HK}) \Rightarrow (\mathrm{E3})_F$.

Proof. Assume $(\mathrm{E3})_F$ and (IC) and let $\mathbf{X} \subseteq \mathbf{P}^T$ and $y \in P^T - X$ where T is non-empty finite. Let $\mathbf{Y}$ be the substructure of $\mathbf{P}^T$ generated by $X \cup \{y\}$. By $(\mathrm{E3})_F$ there are distinct $\varphi, \psi \in E(Y)$ which agree on X, and therefore not at y. By (IC), φ and ψ are restrictions of T-ary term functions of $\mathfrak{P}$.

Next, assume (HK) and let $\mathbf{X} \subsetneqq \mathbf{Y} \subseteq \mathbf{P}^T$, where T is finite. Consider any $y \in Y - X$. By (HK) there are σ, $\tau \in F_T$ which agree on X but not at y. By Lemma 2.21, $\pi_Y \sigma$, $\pi_Y \tau \in E(Y)$. This establishes $(\mathrm{E3})_F$.

Now we obtain an adjusted version of the Second Full-Duality Theorem of Davey and Werner [7]:

Corollary 2.36. *Suppose* $\mathbf{POp} = \emptyset$, $\mathbf{Rl}$ *is finite and* (IC) *holds. Then the duality* (D, E) *is full between* $\mathscr{L}$ *and* $\mathscr{R}_0$ *if and only if* $(\mathrm{E3})_F$ *holds.*

Proof. Use Corollary 2.34 and Lemma 2.35.

In those cases where $\mathbf{POp} \neq \emptyset$ we shall obtain Step 2 of the Full Duality Strategy by a completely different approach. A finite non-trivial algebra $\mathfrak{P}$ is called *filtral* if all congruences on subdirect products of subalgebras of $\mathfrak{P}$ are induced by filters on the index set. Using Jónsson [14] it is not hard to verify that $\mathfrak{P}$ is filtral if and only if it generates a congruence distributive variety and its non-trivial subalgebras are all simple (cf. KRAUSS [15]). So let $\mathfrak{P}$ be a filtral algebra, let K be the set of non-trivial isomorphisms between non-trivial subalgebras of $\mathfrak{P}$ and let E be the set of elements of P which determine a trivial subalgebra of $\mathfrak{P}$. Consider $\mathbf{P}=\langle P, \eta, e\rangle_{\eta\in K, e\in E}$.

Theorem 2.37. $\mathbf{S}_{hk}\mathbf{P}\mathbf{P}=\mathbf{S}_c\mathbf{P}\mathbf{P}$.

Proof. Let $\mathbf{X}\subseteq\mathbf{P}^S$ be closed, $\mathfrak{A}\subseteq\mathfrak{P}^I$ and $f\colon \mathfrak{A}\to\mathfrak{P}$. By Lemma 2.15 we must show that X is closed under $\bar{f}$. So consider $x\in X^I$, where $\pi_s\circ x\in A$ for all $s\in S$. We have to show $\bar{f}(x)\in X$, where $\bar{f}(x)(s)=f(\pi_s\circ x)$. If f has constant value $e\in E$ then $f(x)=\bar{e}\in X$, since $\mathbf{X}\subseteq\mathbf{P}^S$. Otherwise, since $\mathfrak{P}$ is filtral, there is an ultrafilter U on I such that for $a, b\in A$,

$$f(a)=f(b) \quad\text{iff}\quad \operatorname{Eq}(a,b)\in U, \quad h(a)=p \quad\text{iff}\quad a^{-1}(p)\in U.$$

Since f and h have the same kernel, there exists an isomorphism $\eta\colon h(\mathfrak{A})\Vert\!\!\twoheadrightarrow f(\mathfrak{A})$ such that $f=\eta\circ h$. It follows that $\bar{f}=\eta^{\mathbf{X}}\circ\bar{h}$, where $\eta^{\mathbf{X}}$ is the canonical extension of η to X. Since $\mathbf{X}\subseteq\mathbf{P}^S$, X is closed under $\eta^{\mathbf{X}}$, so that $\bar{f}(s)\in X$ iff $\bar{h}(x)\in X$. Since X is closed, for $\bar{h}(x)\in X$ it suffices that for any finite $T\subseteq S$, $\pi_T\bar{h}(s)\in\pi_T X$. For each $t\in T$ set

$$\bar{h}(x)(t) = h(\pi_t\circ x) = p_t\in P,$$

and let

$$M = \bigcap_{t\in T}(\pi_t\circ x)^{-1}(p_t).$$

Then $M\in U$. Consider any $i\in M$. Then for any $t\in T$,

$$x_i(t) = (\pi_t\circ x)(i) = p_t = \bar{h}(x)(t).$$

This establishes that $\pi_T\bar{h}(x)\in\pi_T X$.

Gathering what we have found we can now state a very general result with many immediate applications. If the finite nontrivial algebra $\mathfrak{P}$ has a $k+1$-ary *near unanimity term* we perform *Step 1* by taking all members of $\mathbf{S}\mathfrak{P}^k$ as relations for $\mathbf{P}$ according to Corollary 2.28. Now MITSCHKE [18] has shown that such an algebra always generates a congruence distributive variety. Thus $\mathbf{P}$ is also filtral just in case its *non-trivial subalgebras are all simple.* In this case we can do *Step 2* by adding the set K of non-trivial isomorphisms between non-trivial subalgebras of $\mathfrak{P}$ as partial operations of $\mathbf{P}$ and the set E of elements which determine a trivial subalgebra of $\mathfrak{P}$ as constants of $\mathbf{P}$ according to Theorem 2.37. *Step 3* remains as a clean-up operation which uses Lemma 2.31 and relies on more special properties of the algebra $\mathfrak{P}$.

Theorem 2.38. *Suppose $\mathfrak{P}$ has a $k+1$-ary near unanimity term and only simple non-trivial subalgebras. Let F, G and R be sets of operations, partial operations and* *relations on P respectively, where $E \subseteqq F$, $K \subseteqq G$ and $R \subseteqq S\mathfrak{P}^k$. If*

$$\boldsymbol{P} = \langle P, r, g, f\rangle_{r\in R, g\in G, f\in F}$$

satisfies the conditions of Lemma 2.1 and $R \cup G \cup F$ generates $S\mathfrak{P}^k$, then (D, E) is a full duality between $\mathbf{ISP}\mathfrak{P}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$.

A non-trivial finite algebra $\mathfrak{P}$ is called a dual discriminator algebra if the dual discriminator on P, defined by

$$d^P(x, y, z) = \begin{cases} x & \text{if} \quad x = y \\ z & \text{if} \quad x \neq y \end{cases}$$

is a term function of $\mathfrak{P}$. This notion is due to FRIED and PIXLEY [9]. In this case the dual discriminator serves as a 3-ary near unanimity term on $\mathfrak{P}$ and forces nontrivial subalgebras to be simple, so Theorem 2.38 applies.

Corollary 2.39. *If $\mathfrak{P}$ is a dual discriminator algebra and*

$$\boldsymbol{P} = \langle P, r, \eta, e\rangle_{r\in S\mathfrak{P}^2,\, \eta\in K,\, e\in E}$$

then (D, E) is a full duality between $\mathbf{ISP}\mathfrak{P}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$.

DAVEY and WERNER [7] give many examples of full duality applying their Second Full Duality Theorem. Three of these applications are not correct (quasi primal algebras, weakly associative lattices and median algebras) because in those cases $\boldsymbol{POp} \neq \emptyset$. These erroneous arguments also appear in WERNER [23]. Now it turns out that all three examples are dual discriminator varieties and we can still establish their claims as consequences of Theorem 2.38. As additional applications of Theorem 2.38 we consider primal algebras, quasi primal algebras, distributive lattices (where $\mathfrak{P}$ is a dual discriminator algebra which is not quasi primal) and De Morgan algebras (where $\mathfrak{P}$ is a filtral near unanimity algebra which is not dual discriminator). Finally we give an application of Corollary 2.34 considering semi lattices with unit (where $\mathfrak{P}$ is neither filtral nor near unanimity). The reader will easily convince himself that the remaining examples in DAVEY and WERNER [7] can be treated similarly.

Example 2.40: Primal algebras. A non-trivial finite algebra $\mathfrak{P}$ is called *primal* if every (finitary) operation on $\mathfrak{P}$ is a term function. Clearly a primal algebra has a 3-ary near-unanimity term. Moreover, it is simple and has neither proper subalgebras nor nontrivial automorphisms. Let $\boldsymbol{P} = \langle P\rangle$, i.e. take $\boldsymbol{Op} \cup \boldsymbol{POp} \cup \boldsymbol{Rl} = \emptyset$.

Theorem 2.41. *If $\mathfrak{P}$ is a primal algebra, then (D, E) is a full duality between* $\mathbf{ISP}\mathfrak{P}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$.

Proof. $H \cup E = \emptyset$ and the subalgebras of $\mathfrak{P}^2$ are $\mathfrak{P}^2$ and the diagonal of $\mathfrak{P}^2$. It follows at once that $\emptyset$ generates $S\mathfrak{P}^2$. The assertion follows from Theorem 2.38.

Example 2.42: Quasi primal algebras. A non-trivial finite algebra $\mathfrak{Q}$ is called *quasi primal* if the *ternary discriminator* on Q, defined by

$$t^Q(x, y, z) = \begin{cases} x & \text{if} \quad x \neq y \\ z & \text{if} \quad x = y \end{cases}$$

is a term function of $\mathfrak{Q}$. Since $d^Q(x, y, z) = t^Q(x, t^Q(x, y, z), z)$, a quasi primal algebra is a dual discriminator algebra and Theorem 2.38 applies. To carry out Step 3 of the Full Duality Strategy in this case, several options appear to be available and we choose the setting of DAVEY and WERNER [7]. Let H be the set of *all* isomorphisms between *all* subalgebras of $\mathfrak{Q}$ together with the empty mapping and let $\mathbf{Q} = \langle Q, \eta, e \rangle_{\eta \in H,\, e \in E}$.

Theorem 2.43. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{Q}$ *and* $\mathbf{IS}_c\mathbf{PQ}$.

Proof. The subalgebras of $\mathfrak{Q}^2$ are exactly the direct products of subalgebras of $\mathfrak{Q}$ and the isomorphisms between subalgebras of $\mathfrak{Q}$. It easily follows that $H \cup E$ generates $\mathbf{S}\mathfrak{Q}^2$. The assertion follows from Theorem 2.38.

Example 2.44: Distributive lattices. Let $\mathfrak{D} = \langle \{0, 1\}, \wedge, \vee \rangle$ be the two-element lattice generating the (quasi) variety of distributive lattices. Then

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

defines the dual discriminator on $\{0, 1\}$ and Theorem 2.38 applies. Let $\mathbf{D} = \langle \{0, 1\}, \leqq, 0, 1 \rangle$.

Theorem 2.45. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{D}$ *and* $\mathbf{IS}_c\mathbf{PD}$.

Proof. $\mathfrak{D}$ has no non-trivial automorphisms and it is easy to verify that $\{\leqq\} \cup \cup \{0, 1\}$ generates $\mathbf{S}\mathfrak{D}^2$. The assertion follows from Theorem 2.38.

Example 2.46: Weakly associative lattices. $\mathfrak{W} = \langle W, \wedge, \vee \rangle$ is called a *weakly associative lattice* if it satisfies the lattice axioms with the exception that the associative laws are replaced by the weak associative laws

$$((x \wedge z) \vee (y \wedge z)) \vee z = z, \quad ((x \vee z) \wedge (y \vee z)) \wedge z = z.$$

This notion is due to FRIED and GRÄTZER [8]. A weakly associative lattice has the *unique bound property* if any two elements have unique upper and lower bounds. FRIED and PIXLEY [9] show that a non-trivial finite weakly associative lattice is a filtral algebra if and only if it is a dual discriminator algebra if and only if it has the unique bound property. So let $\mathfrak{W} = \langle W, \wedge, \vee \rangle$ be a non-trivial finite weakly associative lattice with the unique bound property. Then Theorem 2.38 applies. Let H be the set of *all* isomorphisms between *all* subalgebras of $\mathfrak{W}$ together with the empty mapping, let $\leqq$ be the ordering on any fixed two-element subalgebra of $\mathfrak{W}$ and let $\boldsymbol{W}=\langle W, \leqq, \eta, e\rangle_{\eta\in H,\, e\in E}$.

Theorem 2.47. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{W}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{W}$.

Proof. WERNER [23] shows that $\{\leqq\}\cup H\cup E$ generates $\mathbf{S}\mathfrak{W}^2$. The assertion follows from Theorem 2.38.

Example 2.48: Median algebras. Let $\mathfrak{M}_2=\langle\{0, 1\}, d\rangle$, where d is the dual discriminator on $\{0, 1\}$. Then Theorem 2.38 applies. The only automorphism of $\mathfrak{M}_2$ is defined by $0'=1$ and $1'=0$. Let $\boldsymbol{M}_2=\langle\{0, 1\}, \leqq, ', 0, 1\rangle$.

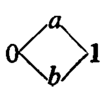
Theorem 2.49. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{M}_2$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{M}_2$.

Proof. WERNER [23] shows that $\{\leqq\}\cup\{'\}\cup\{0, 1\}$ generates $\mathbf{S}\mathfrak{M}_2^2$. The assertion follows from Theorem 2.38.

Example 2.50: DeMorgan algebras. The (quasi) variety of DeMorgan algebras is generated by the algebra $\mathfrak{M}=\langle\{0, a, b, 1\}, \wedge, \vee, 0, 1, \sim\rangle$ where $\langle\{0, a, b, 1\}, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with a and b incomparable and $\sim$ is the unary operation defined by $\sim 0=1$, $\sim 1=0$, $\sim a=a$, $\sim b=b$. It is easy to check that $\mathfrak{M}$ has only simple subalgebras and

$$m(x, y, z) = (x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$$

is a 3-ary near unanimity term for $\mathfrak{M}$. This time $m^{\mathfrak{M}}$ is not the dual discriminator on $\mathfrak{M}$, and BLOK and PIGOZZI [4] check that the dual discriminator is not a term function of $\mathfrak{M}$ at all. Now let $\leqq$ be the partial ordering



on $\{0, a, b, 1\}$ and let α be the automorphism of $\mathfrak{M}$ that interchanges a and b. Let $\boldsymbol{M}=\langle\{0, a, b, 1\}, \leqq, \alpha\rangle$.

Theorem 2.51. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{M}$ *and* $\mathbf{IS}_c\mathbf{P}\boldsymbol{M}$.

Proof. DAVEY and WERNER [7] verify that $\{\leqq\}\cup\{\alpha\}$ generates all 45 subalgebras of $\mathfrak{M}^2$. The assertion follows from Theorem 2.38.

Example 2.52: Semi lattices with unit. Let $\mathfrak{S}=\langle\{0, 1\}, \wedge, 1\rangle$ be the two-element semi lattice generating the (quasi) variety of semi lattices with unit. Now take $\boldsymbol{S}=\langle\{0, 1\}, \wedge, 1\rangle$.

Theorem 2.53. *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{S}$ *and* $\mathbf{IS}_c\mathbf{P}\bar{\boldsymbol{S}}$.

Proof. DAVEY and WERNER [7] verify (IC) and $(E3)_F$, and we verify (HK) directly: Choose T finite, $\mathbf{X} \subseteq \mathbf{S}^T$, $y \in S^T - X$. Let

$$z = \bigwedge \{x \in X \mid x \geqq y\}.$$

Now define $\sigma, \tau \in F_T$, by

$$\sigma = \bigwedge \{\pi_t \mid z(t) = 1\}, \quad \tau = \bigwedge \{\pi_t \mid y(t) = 1\}.$$

Since $y \notin X$, $z > y$ so there exists $t \in T$ such that $z(t)=1$ and $y(t)=0$. Thus $\sigma(y)=0$ but $\tau(y)=1$. However for any $x \in X$, if $x \geqq y$ then $x \geqq z$ so that $\sigma(x)=\tau(x)=1$, whereas if $x \not\geqq y$ then $x \not\geqq z$ so that $\sigma(x)=\tau(x)=0$. This establishes (HK). Now use Corollary 2.34.

3. Axioms for topological quasi atomical theories

In this section we shall present examples of (compact) topological quasi varieties and investigate their topological quasi atomical theories.

Example 3.1. A topological space is T_0 if distinct points have distinct neighborhood systems. Consider the empty similarity type $\mathbf{t}$, i.e. $\boldsymbol{Op} \cup \boldsymbol{POp} \cup \boldsymbol{Rl} = \emptyset$, and let $P_0 = \{0, 1\}$ be the *Sierpinski space* with open sets $\emptyset$, $\{0\}$, $\{0, 1\}$. Then for any topological space $\boldsymbol{Y}$ the following are equivalent:

(i) $\boldsymbol{Y} \in \mathbf{ISPP}_0$,

(ii) $\boldsymbol{Y} \models [u_d \xrightarrow{d \in \{0\}} v_0 \wedge v_d \xrightarrow{d \in \{0\}} u_0] \Rightarrow u \approx v$,

(iii) $\boldsymbol{Y}$ is a T_0-space.

Proof. The equivalence of (ii) and (iii) is obvious and (i) implies (ii) by Corollary 1.10. Finally assume (iii). We shall verify conditions (i) and (iv) of the Separation Principle. Suppose $x, y \in Y$, where $x \neq y$. Then, say, there exists an open neighborhood U of x, where $y \notin U$. Define

$$\varphi(z) = \begin{cases} 0 & \text{if } z \in U, \\ 1 & \text{if } z \notin U. \end{cases}$$

This establishes condition (i) of the Separation Principle. Suppose D is a subset of the power set of Y directed by inclusion, $\delta \colon D \to Y$ is a net in Y and $y \in Y$, where $\delta d \overset{d \in D}{\nrightarrow} y$. Then there exists an open neighborhood U of Y such that δ is not eventually in U. Now define φ as before and condition (iv) of the Separation Principle is established. (i) of the assertion now follows from the Separation Principle.

This example shows that the class of T_0-*spaces* is a topological quasi variety in *any* similarity type.

Example 3.2. A topological space is Hausdorff if and only if limits of convergent nets are unique. Thus a topological structure $\mathbf{X}$ (of any similarity type $\mathbf{t}$!) is Hausdorff if and only if for any directed set $\langle D, \leqq\rangle$ and any net v: $D \to Vb$, $\mathbf{X}$ is a model of

$$[v_d \xrightarrow{d\in D} u \wedge v_d \xrightarrow{d\in D} w] \Rightarrow u \approx w.$$

Thus the class of topological *Hausdorff* structures is a topological quasi variety in *any* similarity type. While our axioms are simple and uniform they constitute a proper class, and we claim that this is essential.

Lemma 3.3. *For each regular cardinal $\varkappa$ there is a compact topological space $X_\varkappa$ which is not Hausdorff but whose subspaces of cardinality less than $\varkappa$ are zero-dimensional (i.e. have a basis of clopen sets) Hausdorff.*

Proof. For $X_\varkappa$ take the set $\varkappa+2$ with subbasis consisting of all sets $U\cup\{\varkappa\}$ and $U\cup\{\varkappa+1\}$ where $U\subseteqq\varkappa$ is an open interval.

Corollary 3.4. *Let Σ be the topological quasi-atomical theory of Hausdorff spaces. Then every subset Σ_0 of Σ has a model which is not Hausdorff.*

Proof. Let $\varkappa$ be a regular (e.g., successor) cardinal larger than the number of variables occurring in the formulas of Σ_0. We verify that $\mathbf{X}_\varkappa \models \Sigma_0$. Let $\Phi\in\Sigma_0$ and choose a: $Vb \to X_\varkappa$. Let $\mathbf{Y}$ be the subspace of $\mathbf{X}_\varkappa$ determined by the images under a of the variables that occur in Φ. Without loss of generality we may assume that a: $Vb\to Y$. Since $\mathbf{Y}$ has smaller cardinality than $\varkappa$, $\mathbf{Y}\models\Phi[a]$ by Lemma 3.3. It follows from Lemma 1.9 (i) that $\mathbf{X}_\varkappa\models\Phi[a]$.

Example 3.5. Consider the empty similarity type $\mathbf{t}$, i.e. $\mathbf{Op}\cup\mathbf{POp}\cup\mathbf{Rl}=\emptyset$, and let P be a finite set with at least two elements carrying the *discrete topology*. Then for any topological space $\mathbf{Y}$ the following are equivalent:

(i) $\mathbf{Y}\in\mathbf{ISPP}$,

(ii) $\mathbf{Y}\models\mathrm{Th}_{\mathrm{tqa}}\mathbf{P}$,

(iii) $\mathbf{Y}$ is a zero-dimensional Hausdorff space.

Proof. The equivalence of (i) and (ii) follows from Corollary 1.10, and (i) implies (iii) trivially. Finally assume (iii). We shall verify conditions (i) and (iv) of the Separation Principle. Suppose $x, y\in Y$, where $x\neq y$. Since $\mathbf{Y}$ is a zero-dimensional Hausdorff space, there exists a clopen neighborhood U of x, where $y\notin U$. Choose $a, b\in P$, where $a\neq b$ and define

$$\varphi(z)=\begin{cases} a & \text{if } z\in U,\\ b & \text{if } z\notin U.\end{cases}$$

This establishes condition (i) of the Separation Principle. To establish condition (iv) we proceed just as in Example 3.1.

This example shows that the class of *totally disconnected Hausdorff* structures is a topological quasi variety in *any* similarity type.

The difference between Examples 3.1 and 3.2 on the one side and Example 3.5 on the other forces us to broach the subject of "axiomatizability" of a topological quasi atomical theory. Intuitively speaking, a theory is axiomatizable in case it is "intelligible" and, roughly speaking, this means that axioms for the theory can be "explicitly written down" in some sense. Well-known technical expliciations of this notion then follow suit, which in the case of infinitary languages (like ours!) require rather sophisticated machinery. Without getting involved in all of this (and we shall not!), it is clear that in Examples 3.1 and 3.2 we have explicitly written down axioms for the topological quasi atomical theories of T_0-*spaces* and of *Hausdorff* spaces respectively, whereas in Example 3.5 we have *not* explicitly written down axioms for the theory of *zero-dimensional Hausdorff* spaces. The reason is simple: We have not been able to. Since further inquiries into this matter require considerations going beyond the scope of this paper, we shall leave it at that.

Example 3.6. Let $\mathbf{t}$ be the similarity type of topological Abelian groups $\langle G, +, -, 0\rangle$, and let $\mathbf{C}$ be the circle group of real numbers modulo the integers with the quotient topology. By Pontryagin's Duality (PONTRYAGIN [21]), for any compact topological structure $\mathbf{Y}$, $\mathbf{Y} \in \mathbf{IS}_c\mathbf{PC}$ if and only if $\mathbf{Y}$ is a compact Abelian group.

Of course, the axioms for Abelian groups (trivially) "axiomatize" the topological quasi atomical theory of topological Abelian groups. However, that the class of compact Abelian groups is generated (as a compact topological quasi variety) by $\mathbf{C}$ is a highly nontrivial observation.

Now we shall turn to the examples of Section 2. Each "full duality result" obtained from Corollaries 2.34, 2.36 and Theorem 2.38 yields two category anti-equivalences

$$D: \mathbf{ISP}\mathfrak{P} \to \mathbf{IS}_c\mathbf{P}\boldsymbol{P}, \quad E: \mathbf{IS}_c\mathbf{P}\boldsymbol{P} \to \mathbf{ISP}\mathfrak{P}$$

between the (algebraic) quasi variety $\mathbf{ISP}\mathfrak{P}$ and the compact (topological) quasi variety $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$ which are inverse to each other (Lemma 2.10). The primary goal of this kind of "unique representation" is to gain insight into the quasi variety $\mathbf{ISP}\mathfrak{P}$ from ones knowledge of the compact quasi variety $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$ (at least this appears to be the original motivation for "duality results"!). An obvious prerequisite for success is that one "knows" which topological structures $\boldsymbol{X}$ belong to $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$. Now in a sense one does because the definition of the quasi variety $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$ contains a clear description of its members. However, this description is not very helpful to "decide" whether a given topological structure $\boldsymbol{X}$ does belong to $\mathbf{IS}_c\mathbf{P}\boldsymbol{P}$ or not. In fact the Compact Hausdorff Separation Principle (Corollary 1.3), which really just spells out the description "topological isomorph of a compact substructure of a power of $\boldsymbol{P}$", rarely is helpful in this task. What is needed is a description of the members of

$\mathbf{IS}_c\mathbf{PP}$ in terms of their topology and structure which does not involve "constructions". Now in Section 1 we have done exactly that. We have found the "right" language to characterize the members of $\mathbf{IS}_c\mathbf{PP}$ as the *compact models of the topological quasi atomical theory of* $\mathbf{P}$. But in general this is not an "intelligible" description in the sense that it is not possible to "decide" whether a given topological quasi atomic formula Φ of this language does belong to $\mathrm{Th}_{\mathrm{tqa}}\mathbf{P}$ or not. What is required is an "*axiomatization*" *of the topological quasi atomical theory* of $\mathbf{P}$. Now we have already run into trouble with that task failing to axiomatize the topological quasi atomical theory of zero dimensional Hausdorff spaces in Example 3.5. Fortunately what is required in the examples arising in the setting of Section 2 is something more special: We need to find an "axiomatization" of the topological quasi atomical theory of the *compact* topological quasi variety $\mathbf{IS}_c\mathbf{PP}$, and this we can do.

Example 3.7: Boolean spaces. Returning to the setting of Example 3.5 we show that we can axiomatize the compact topological quasi variety $\mathbf{IS}_c\mathbf{PP}$ of Boolean spaces. Let BL denote the class of all formulas

$$\{\Phi \mid \Phi \in \Sigma\} \Rightarrow x \approx y$$

where $\mathbf{Y}$ is a compact space, Σ is defined as in Lemma 1.11 and x and y are two points of $\mathbf{Y}$ which are not separated by clopen sets. Then for any topological space $\mathbf{Y}$ the following are equivalent:

(i) $\mathbf{Y} \in \mathbf{IS}_c\mathbf{PP}$;
(ii) $\mathbf{Y}$ is a compact model of BL;
(iii) $\mathbf{Y}$ is a Boolean space.

Proof. The equivalence of (i) and (iii) follows from Example 3.5. Next assume (iii). To prove (ii) it is sufficient, by Lemma 1.10 and (i), to verify that $\mathbf{P} \models BL$. Accordingly consider

$$\{\Phi \mid \Phi \in \Sigma\} \Rightarrow x \approx y$$

in BL as above, $b\colon Vb \to P$ such that $\mathbf{P} \models \Phi[b]$ for each $\Phi \in \Sigma$. By Lemma 1.11 $\varphi\colon \mathbf{Y} \to \mathbf{P}$, where $\varphi(v) = b(v)$. If $bx \neq by$, then $\varphi^{-1}(bx)$ and $\varphi^{-1}(by)$ are clopen sets separating x and y. Thus $bx = by$ and therefore $\mathbf{P} \models (x \approx y)[b]$.

Conversely assume (iii) fails. Since $\mathbf{Y}$ is compact there must be distinct members x and y of $\mathbf{Y}$ which are not separated by clopen sets. But then

$$\{\Phi \mid \Phi \in \Sigma\} \Rightarrow x \approx y$$

is a formula of BL not satisfied by $\mathbf{Y}$, and we conclude that (ii) fails.

As in Example 3.2, the size of the axiom system cannot be reduced.

Corollary 3.8. *Let Σ be the topological quasi-atomical theory of Boolean spaces. Then every subset Σ_0 of Σ has a compact model that is not Boolean.*

Proof. The proof is exactly the same as in Corollary 3.4 since every zero-dimensional Hausdorff space can be embedded in a Boolean space.

The axiom system BL will be incorporated into our subsequent examples. Before we continue we should like to make a few comments on the nature of this axiomatization of the topological quasi atomical theory of Boolean spaces. Although it appears to be a "reasonable" axiomatization of the (infinitary) *first-order* theory under consideration, it does not appear to be a *mathematically useful* characterization of Boolean spaces. Now the usual definition of Boolean spaces obviously translates into a simple *higher order* definition in the language under consideration. Thus item (ii) of Example 3.7 may be viewed as a (mathematically useless) *first-order* axiomatization of the compact quasi variety $\mathbf{IS}_c\mathbf{PP}$, whereas item (iii) may be viewed as a (mathematically useful) *higher-order* axiomatization. This pattern will reoccur in all subsequent examples.

Corollary 3.9 (Hu [13]). *If* $\mathfrak{P}$ *is a primal algebra, then* (D, E) *is a full duality between* $\mathbf{ISP}\mathfrak{P}$ *and the category of Boolean spaces.*

Proof. Use Theorem 2.41 and Example 3.7.

Example 3.10: Boolean H-spaces. Return to the setting of Example 2.42. Actually, we shall consider a more general setting where $\mathfrak{Q}$ is an arbitrary non-trivial finite algebra, H is the set of *all* isomorphisms between *all* subalgebras of $\mathfrak{Q}$ together with the empty mapping, and E is the set of elements of Q determining a trivial subalgebra of $\mathfrak{Q}$.

Now take $\mathbf{Q}=\langle Q, \eta, e\rangle_{\eta\in H,\, e\in E}$ and consider each $\eta\in H$ as a partial operation symbol, and each $e\in E$ as an individual constant to determine the similarity type $\mathbf{t}$. Davey and Werner [7] call a topological $\mathbf{t}$-structure $\mathbf{X}=\langle X, \eta^{\mathbf{X}}, e^{\mathbf{X}}\rangle_{\eta\in H,\, e\in E}$ a *Boolean H-space* if

(i) $\mathbf{X}$ is a Boolean space.

(ii) Each $\eta^{\mathbf{X}}$ is a homeomorphism between closed subspaces of $\mathbf{X}$.

(iii) $(\eta\circ\gamma)^{\mathbf{X}}=\eta^{\mathbf{X}}\circ\gamma^{\mathbf{X}}$.

(iv) $(\eta\cap\gamma)^{\mathbf{X}}=\eta^{\mathbf{X}}\cap\gamma^{\mathbf{X}}$.

(v) If η is the identity on Q, then $\eta^{\mathbf{X}}$ is the identity on X.

(vi) $\emptyset^{\mathbf{X}}=\emptyset$.

(vii) If $e\in E$ and η is the identity on $\{e\}$, then $\eta^{\mathbf{X}}$ is the identity on $\{e^{\mathbf{X}}\}$.

(viii) If $e_0, e_1\in E$ and $\eta e_0=e_1$, then $\eta^{\mathbf{X}}e_0^{\mathbf{X}}=e_1^{\mathbf{X}}$.

We need a topological fact whose verification is straightforward.

Lemma 3.11. *Let X and Y be compact Hausdorff spaces,* $X_0\subseteq X$ *and* $g\colon X_0\to Y$. *Then (the graph of) g is closed in* $X\times Y$ *if and only if* X_0 *and* $g(X_0)$ *are both closed and g is continuous.*

Using this lemma we observe that the definition of Boolean H-space translates directly into topological quasi equational axioms:

Lemma 3.12. *Let $\mathfrak{Q}$ be a nontrivial finite algebra. Then a compact $\mathbf{t}$-structure $\mathbf{X}$ is a Boolean H-space if and only if it is a model of*

(i)′ *BL.*

(ii)′ $\mathrm{Cl}(\eta)$, $\eta u\approx\eta v\Rightarrow u\approx v$, *where* $\eta\in H$ *and* $\mathrm{Cl}(\eta)$ *is defined in Example 1.6.*

(iii)′ $\eta\delta u\approx v\Leftrightarrow\gamma u\approx v$, *where* $\eta,\delta,\gamma\in H$ *and* $\eta\circ\delta=\gamma$.

(iv)′ $\gamma u\approx v\Leftrightarrow[\eta u\approx v\wedge\delta u\approx v]$, *where* $\gamma,\eta,\delta\in H$ *and* $\gamma=\eta\cap\delta$.

(v)′ $\eta v\approx v$, *where η is the identity on Q.*

(vi)′ $\emptyset u\not\approx v$.

(vii)′ $\eta u\approx v\Leftrightarrow[u\approx e\wedge v\approx e]$, *where η is the identity on* $\{e\}$.

(viii)′ $\eta e_0\approx e_1$, *where* $e_0,e_1\in E$ *and* $\eta e_0=e_1$.

Proof. The equivalence of (i) and (i)′ is Example 3.7 and the equivalence of (ii) and (ii)′ follows from Lemma 3.11. The remaining equivalences are straightforward.

Our goal is to prove that the topological quasi variety generated by $\mathbf{Q}$ is exactly the class of Boolean H-spaces. This will require a somewhat detailed examination of the consequences of the axioms (i)′–(viii)′ for Boolean H-spaces. While some parts of our argument may be found in Davey and Werner's proof of SEP, a correct proof of the Compact Hausdorff Separation Principle appears to require that we reproduce our argument in full. To do so we introduce some more specialized notation. If $\mathfrak{A}\subseteq\mathfrak{Q}$ and $\mathbf{X}$ is a Boolean H-space, we let "1_A" denote the identity on A and "X_A" the domain of $1_A^{\mathbf{X}}$. Moreover, we write "$1_\emptyset$" for the empty map: $1_\emptyset=\emptyset\in H$, and "$X_\emptyset$" for the domain of $1_\emptyset^{\mathbf{X}}$: $X_\emptyset=\emptyset\subseteq X$.

Lemma 3.13. *Let $\mathbf{X}$ be a Boolean H-space, $\eta\in H$, $\mathfrak{A}$ and $\mathfrak{B}$ subalgebras of $\mathfrak{Q}$.*

(i) $1_A^{\mathbf{X}}$ *is the identity on* X_A.

(ii) *If η has domain A, then $\eta^{\mathbf{X}}$ has domain X_A.*

(iii) $X_{A\cap B}=X_A\cap X_B$.

(iv) $\mathfrak{A}\subseteq\mathfrak{B}$ implies $X_A\subseteq X_B$.

(v) $(\eta^{-1})^{\mathbf{X}}=(\eta^{\mathbf{X}})^{-1}$.

Proof. (i) $1_A\cap 1_Q=1_A$ so $1_A^{\mathbf{X}}\cap 1_Q^{\mathbf{X}}=1_A^{\mathbf{X}}\cap 1_X=1_A^{\mathbf{X}}$. Thus $1_A^{\mathbf{X}}\subseteq 1_X$.

(ii) Let $Y\subseteq X$ be the domain of $\eta^{\mathbf{X}}$. $\eta\circ 1_A=\eta$ so $\eta^{\mathbf{X}}\circ 1_A^{\mathbf{X}}=\eta^{\mathbf{X}}$ so $Y\subseteq X_A$. Then $\eta^{-1}\circ\eta=1_A$, so $(\eta^{-1})^{\mathbf{X}}\circ\eta^{\mathbf{X}}=1_A^{\mathbf{X}}\supseteq 1_Y$. It follows that $1_A^{\mathbf{X}}=1_Y$ so $X_A=Y$.

(iii) $X_{A\cap B}=\mathrm{dom}\,1_{A\cap B}^{\mathbf{X}}=\mathrm{dom}\,(1_A\cap 1_B)^{\mathbf{X}}=\mathrm{dom}\,(1_A^{\mathbf{X}}\cap 1_B^{\mathbf{X}})=X_A\cap X_B$.

(iv) Use (iii).

(v) Let η have domain A. Then $\eta^{-1}\circ\eta=1_A$ so $(\eta^{-1})^{\mathbf{X}}\circ\eta^{\mathbf{X}}=1_A^{\mathbf{X}}$. It follows that $(\eta^{\mathbf{X}})^{-1}\subseteq(\eta^{-1})^{\mathbf{X}}$. Replacing η by η^{-1}, we obtain $[(\eta^{-1})^{\mathbf{X}}]^{-1}\subseteq\eta^{\mathbf{X}}$ so $(\eta^{-1})^{\mathbf{X}}\subseteq(\eta^{\mathbf{X}})^{-1}$.

Lemma 3.14 (DAVEY and WERNER [7], 2.7, (3)). $\mathbf{Q}$ *is injective in the category of Boolean H-spaces.*

Our proof of the Compact Hausdorff Separation Principle will depend on finding "many" continuous homomorphisms from a Boolean H-space $\mathbf{X}$ into $\mathbf{Q}$. We say $x\in X$ is a *fixed point* of $\eta^{\mathbf{X}}$, $\eta\in H$, if x is in the domain of $\eta^{\mathbf{X}}$ and $\eta^{\mathbf{X}}x=x$. Now define

$$H_x=\{\eta\in H \mid x \text{ is a fixed point of } \eta^{\mathbf{X}}\}$$

and let $\mathfrak{S}_x$ be the subalgebra of $\mathfrak{Q}$ consisting of all elements fixed by each member of H_x. Notice that every morphism $\varphi\colon \mathbf{X}\to\mathbf{Q}$ takes x into S_x. Let $E^{\mathbf{X}}=\{e^{\mathbf{X}} \mid e\in E\}$ and let $\mathbf{H}x$ be the substructure of $\mathbf{X}$ generated by x:

$$Hx=\{\eta^{\mathbf{X}}x \mid \eta\in H,\ x \text{ in the domain of } \eta\}\cup E^{\mathbf{X}}.$$

Lemma 3.15. *Let $\mathbf{X}$ be a Boolean H-space, $x\in X$.*
(i) $x\in X_A$ *if and only if* $\mathfrak{S}_x\subseteq\mathfrak{A}$.
(ii) *If* $x\notin E^{\mathbf{X}}$, *then* $|S_x|>1$.
(iii) *If* $x=e^{\mathbf{X}}\in E^{\mathbf{X}}$, *then* $S_x=\{e\}$.
(iv) *If* $a\in S_x$, *then there is a* $\varphi\colon \mathbf{H}x\to\mathbf{Q}$ *such that* $\varphi(x)=a$.

Proof. (i) If $x\in X_A$ then x is a fixed point of $1_A^{\mathbf{X}}$ so each element of S_x is a fixed point of 1_A. Thus $S_x\subseteq A$. Conversely, let $S_x\subseteq A$. By definition of S_x,

$$1_{S_x} = 1_Q\cap\bigcap\{\eta|\eta\in H_x\}$$

so, by (iv),

$$1_{S_x}^{\mathbf{X}} = 1_Q^{\mathbf{X}}\cap\bigcap\{\eta^{\mathbf{X}}|\eta\in H_x\}.$$

The right contains (x, x) so the left does as well, and $x\in X_{S_x}$. Now $S_x\subseteq A$ so, by (3.13, iv), $x\in X_{S_x}\subseteq X_A$.

(ii) Since $x\in X_{S_x}$ by (i), and $X_\emptyset=\emptyset$, we obtain $S_x\neq\emptyset$. Suppose $S_x=\{e\}$. Then $e\in E$ so by (3.10, vii) we would have $x\in X_{S_x}=X_{\{e\}}=\{e^{\mathbf{X}}\}\subseteq E^{\mathbf{X}}$.

(iii) Consider $x=e^{\mathbf{X}}\in E^{\mathbf{X}}$. Then $e^{\mathbf{X}}$ is the only fixed point of $1_e^{\mathbf{X}}$ (3.10, vii), so $S_x\subseteq\{e\}$. But if $e^{\mathbf{X}}$ is a fixed point of $\eta^{\mathbf{X}}\colon X_A\to X_B$, then

$$\emptyset \neq 1_e^{\mathbf{X}} = 1_e^{\mathbf{X}}\cap\eta^{\mathbf{X}} = (1_e\cap\eta)^{\mathbf{X}}.$$

Since $\emptyset^{\mathbf{X}}=\emptyset$, $1_e\subseteq\eta$ so $\eta e=e\in S_x$.

(iv) If $x=e^{\mathbf{X}}\in E^{\mathbf{X}}$, then $S_x=\{e\}$ so by (viii) $Hx=E^{\mathbf{X}}$ and we define $\varphi(c^{\mathbf{X}})=c$ for $c\in E$. Now suppose $c, d\in E$ and $\eta^{\mathbf{X}}c^{\mathbf{X}}=d^{\mathbf{X}}$. Then

$$(c^{\mathbf{X}}, d^{\mathbf{X}})\in 1_d^{\mathbf{X}}\circ\eta^{\mathbf{X}}\circ 1_c^{\mathbf{X}} = (1_d\circ\eta\circ 1_c)^{\mathbf{X}} \neq \emptyset.$$

Consequently $1_d\circ\eta\circ 1_c\neq\emptyset$ so $\eta(c)=d$ and $\varphi(\eta^{\mathbf{X}}c^{\mathbf{X}})=\varphi(d^{\mathbf{X}})=d=\eta(c)=\eta(\varphi c^{\mathbf{X}})$.

Next consider $x \notin E^{\mathbf{X}}$. Then $\{\eta^{\mathbf{X}}x \mid \eta \in H\} \cap E^{\mathbf{X}} = \emptyset$ so that, in view of the preceding paragraph it remains to find an H-preserving map

$$\psi: \{\eta^{\mathbf{X}}x \mid \eta \in H\} \to Q.$$

If η has domain A and $x \in X_A$, the domain of $\eta^{\mathbf{X}}$, then by (i) $a \in A$. Moreover, if $\eta^{\mathbf{X}}x = \gamma^{\mathbf{X}}x$ then x is a fixed point of $(\gamma^{\mathbf{X}})^{-1} \circ \eta^{\mathbf{X}} = (\gamma^{-1} \circ \eta)^{\mathbf{X}}$, by (3.13, v), so a is a fixed point of $\gamma^{-1} \circ \eta$ and $\eta a = \gamma a$. It follows that ψ is well defined by $\psi(\eta^{\mathbf{X}}x) = \eta a$.

Theorem 3.16. *Let $\mathfrak{Q}$ be a nontrivial finite algebra,* $\mathbf{Q} = \langle Q, \eta, e \rangle_{\eta \in H,\, e \in E}$. *For a* **t**-*structure* $\mathbf{X}$ *the following are equivalent:*

(i) $\mathbf{X} \in \mathbf{IS}_c\mathbf{PQ}$.

(ii) $\mathbf{X}$ *is a compact model of* (i)$'$–(viii)$'$.

(iii) $\mathbf{X}$ *is a Boolean H-space.*

Proof. (ii) and (iii) are equivalent by Lemma 3.12 and (i) implies (ii) by Corollary 1.10. It remains to show that a Boolean H-space $\mathbf{X}$ satisfies the conditions of the Compact Hausdorff Separation Principle, Corollary 1.3.

We first consider $x, y \in X$ where $x \neq y$. By Lemma 3.14 we need only find $\chi: \mathbf{H}x \cup \mathbf{H}y \to \mathbf{Q}$ separating x and y. If $x, y \in E^{\mathbf{X}}$, then $Hx \cup Hy = E^{\mathbf{X}}$ and we take $\chi e^{\mathbf{X}} = e$ (3.15, iv). Otherwise, assume $x \notin E^{\mathbf{X}}$. If $y \in E^{\mathbf{X}}$ take any $\chi: \mathbf{H}x \to \mathbf{Q}$ (3,15, iv). If $y \notin E^{\mathbf{X}}$ and $Hx \cap Hy = E^{\mathbf{X}}$, take any $\varphi: \mathbf{H}x \to \mathbf{Q}$, $\psi: \mathbf{H}y \to \mathbf{Q}$ and let $\chi = \varphi \cup \psi$.

Finally, suppose $x \notin E^{\mathbf{X}}$, $y \notin E^{\mathbf{X}}$ and $Hx = Hy$. Let $\eta^{\mathbf{X}}x = y$. By (3.15, i) S_x is contained in the domain of η. We claim that some member of S_x is not fixed by η. Indeed if $\eta a = a$ for all $a \in S_x$ then would have $\eta \cap 1_{S_x} = 1_{S_x}$ so $\eta^{\mathbf{X}} \cap 1^{\mathbf{X}}_{S_x} = 1^{\mathbf{X}}_{S_x}$. But $x \in X_{S_x}$ (3.15, i) and $\eta^{\mathbf{X}}x = y \neq x$. Choose $a \in S_x$ so that $\eta a \neq a$. Let $\varphi: \mathbf{H}x \to \mathbf{Q}$ take x to a (3.15, iv). Then $\varphi(y) = \varphi(\eta^{\mathbf{X}}x) = \eta\varphi(x) = \eta a \neq \varphi(x)$.

To verify the second condition, choose x not in the domain $\eta^{\mathbf{X}}$. Let η have domain A. Then $x \notin X_A$. By (3.15, i), $S_x \nsubseteq A$. Choose $a \in S_x - A$ and $\varphi: \mathbf{H}x \to \mathbf{Q}$ taking x to a. By Lemma 3.14 φ extends to a $\psi: \mathbf{X} \to \mathbf{Q}$ where $\psi(x) = a$ is not in the domain of η. As there are no relations we obtain (i) from Corollary 1.3.

Corollary 3.17. *If $\mathfrak{Q}$ is a quasi primal algebra, then (D, E) is a full duality between* $\mathbf{ISP}\mathfrak{Q}$ *and the category of Boolean H-spaces.*

Proof. Use Theorems 2.43 and 3.13.

Example 3.18. Bounded Priestley spaces. Return to the setting of Example 2.44. A topological structure $\langle X, \leqq^{\mathbf{X}} \rangle$ is called a *Priestley space* if X is a partially ordered Boolean space and for each $x, y \in X$, if $x \nleqq^{\mathbf{X}} y$ then there is a clopen increasing set containing x but not y. $\langle X, \leqq^{\mathbf{X}}, 0^{\mathbf{X}}, 1^{\mathbf{X}} \rangle$ is a *bounded Priestley space* if $\langle X, \leqq^{\mathbf{X}} \rangle$ is a Priestley space with bounds $0^{\mathbf{X}}$ and $1^{\mathbf{X}}$. Now for each choice of a bounded partially ordered space $\mathbf{Y}=\langle Y, \leqq^{\mathbf{Y}}, 0^{\mathbf{Y}}, 1^{\mathbf{Y}}\rangle$ where $Y \subseteq Vb$, and any $x, y \in Y$, where $x \not\leqq^{\mathbf{Y}} y$ and every clopen increasing set containing x also contains y, define Σ as in Lemma 1.11 and let ψ be the formula

$$\bigwedge\{\Phi \mid \Phi \in \Sigma\} \Rightarrow x \leqq y.$$

Let BPS denote the class of all such ψ. Then for any topological structure $\mathbf{Y}$ the following are equivalent:

(i) $\mathbf{Y} \in \mathbf{IS}_c\mathbf{PD}$.

(ii) $\mathbf{Y}$ is a compact model of BPS and the axioms for bounded posets.

(iii) $\mathbf{Y}$ is a bounded Priestley space.

Proof. Assume (i). To prove (ii) we must show, by Corollary 1.10, that $\mathbf{D} \models BPS$. Let $\psi \in BPS$ as described above, $b\colon Vb \to D$ where $\mathbf{D} \models \Phi[b]$ for each $\Phi \in \Sigma$. By Lemma 1.11, $\varphi\colon \mathbf{Y} \to \mathbf{D}$ where $\varphi(v)=b(v)$. Since φ is a homomorphism $\varphi^{-1}(1)$ is a clopen increasing set. Now if $\varphi(x)=1$ we conclude $x \in \varphi^{-1}(1)$ so $y \in \varphi^{-1}(1)$. It follows that $\varphi(x) \leqq \varphi(y)$, i.e., $\mathbf{D} \models (x \leqq y)[b]$.

To prove (ii) implies (iii), assume that $\mathbf{Y}$ is a compact bounded partially ordered space that is not a Priestley space. Without loss of generality we assume $Y \subseteq Vb$. Then there are $x, y \in Y$ where $x \not\leqq^{\mathbf{Y}} y$ but every clopen increasing set containing x also contains y. Defining Σ as in Lemma 1.11 we obtain

$$\bigwedge\{\Phi \mid \Phi \in \Sigma\} \Rightarrow x \leqq y$$

in BPS not satisfied by $\mathbf{Y}$. Thus (ii) fails.

Finally, it is easy to see that (iii) gives the conditions of the Compact Hausdorff Separation Principle from which we obtain (i).

Corollary 3.19 (PRIESTLEY [19], [20]). *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{D}$ *and the category of bounded Priestley spaces.*

Proof. Use Theorem 2.45 and Example 3.18.

We shall omit the axiomatization of the topological quasi atomical theories of $\mathbf{IS}_c\mathbf{PW}$ (Example 2.46) and of $\mathbf{IS}_c\mathbf{PM}_2$ (Example 2.48).

Example 3.20: DeMorgan algebras. We first observe that by omitting all references to the bounds 0 and 1, in the previous example we would obtain a system PS of axioms for *all* Priestley spaces $\langle X, \leqq^{\mathbf{X}}\rangle$. Now return to the setting of Example 2.50. Then for any topological structure $\mathbf{Y}$ the following are equivalent:

(i) $\mathbf{Y} \in \mathbf{IS}_c\mathbf{PM}$,

(ii) $\mathbf{Y}$ is a compact model of PS, the poset axioms and

$$\alpha\alpha u \approx u, \quad u \leqq v \Rightarrow \alpha v \leqq \alpha u.$$

Proof. (i) implies (ii) by Corollary 1.10 and DAVEY and WERNER [7] show that the conditions of the Compact Hausdorff Separation Principle follow immediately from (ii).

Corollary 3.21 (CORNISH and FOWLER [5]). *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{M}$ *and the category of all Priestley spaces with an order inverting homeomorphism of order two.*

Proof. Use Theorem 2.51 and Example 3.20.

Example 3.22: Boolean semi lattices with unit. Return to the setting of Example 2.52. Then for any topological structure $\mathbf{Y}=\langle Y, \wedge, 1\rangle$ the following are equivalent:

(i) $\mathbf{Y}\in\mathbf{IS}_c\mathbf{PS}$;

(ii) $\mathbf{Y}$ is a compact model of BL and the axioms for commutative semi lattices with unit.

Proof. By Example 3.7 we only have to show that (ii) implies (i). Under the hypothesis of (ii) DAVEY and WERNER [7] verify the conditions of the Compact Hausdorff Separation Principle for Algebras.

Corollary 3.23 (HOFMANN, MISLOVE and STRALKA [12]). *(D, E) is a full duality between* $\mathbf{ISP}\mathfrak{S}$ *and the category of Boolean semi lattices with unit.*

Proof. Use Theorem 2.53 and Example 3.22.

References

[1] H. ANDRÉKA, P. BURMEISTER and I. NÉMETI, Quasi varieties of partial algebras — A unifying approach towards a two-valued model theory for partial algebras, Preprint, 1980.

[2] K. A. BAKER and A. F. PIXLEY, Polynomial interpolation and the Chinese remainder theorem for algebraic systems, *Math. Z.*, **143** (1973), 165—174.

[3] G. BIRKHOFF, On the structure of abstract algebras, *Proc. Cambridge Phil. Soc.*, **31** (1935), 433—454.

[4] W. J. BLOK and D. PIGOZZI, On the structure of varieties with equationally definable principal congruences. I, Preprint.

[5] W. H. CORNISH and P. R. FOWLER, Coproducts of DeMorgan algebras, *Bull. Austral. Math. Soc.*, **16** (1977), 1—13.

[6] B. A. DAVEY, Topological duality for prevarieties of universal algebras, in: *Studies in Foundations and Combinatorics*, Advances in Mathematics Supplementary Studies, 1, Academic Press (New York, 1978), 433—454.

[7] B. A. DAVEY and H. WERNER, Dualities and equivalences for varieties of algebras, Preprint, 1980.

[8] E. FRIED and G. GRÄTZER, A non-associative extension of the class of distributive lattices, *Pacific J. Math.*, **49** (1973), 59—78.

[9] E. FRIED and A. F. PIXLEY, The dual discriminator function in universal algebra, *Acta Sci. Math.*, **41** (1979), 83—100.

[10] G. GRÄTZER, *Universal Algebra*, 2nd Ed., Springer Verlag (New York—Heidelberg—Berlin, 1979).

[11] G. GRÄTZER and H. LAKSER, A note on the implicational class generated by a class of structures, *Canad. Math. Bull.*, **16** (1973), 603—605.

[12] K. H. HOFMANN, M. MISLOVE and A. STRALKA, *The Pontryagin Duality of Compact 0-Dimensional Semilattices and its Applications*, Lecture Notes in Mathematics 396, Springer Verlag (Berlin—New York, 1974).

[13] T. K. HU, Stone duality for primal algebra theory, *Math. Z.*, **110** (1971), 180—198.

[14] B. JÓNSSON, Algebras whose congruence lattices are distributive, *Math. Scand.*, **21** (1967), 110—121.

[15] P. H. KRAUSS, The structure of filtral varieties, Preprint, 1981.

[16] P. H. KRAUSS and D. M. CLARK, Global Subdirect Products, *Memoirs Amer. Math. Soc.*, **210** (1979).

[17] A. I. MAL'CEV, *Algebraic Systems*, Springer Verlag (Berlin, 1973).

[18] A. MITSCHKE, Near unanimity identities and congruence distributivity in equational classes, *Algebra Universalis*, **8** (1978), 29—32.

[19] H. A. PRIESTLEY, Representation of distributive lattices by means of ordered Stone spaces, *Bull. London Math. Soc.*, **2** (1970), 186—190.

[20] H. A. PRIESTLEY, Ordered topological spaces and the representation of distributive lattices, *Proc. London Math. Soc. (3)*, **24** (1972), 507—530.

[21] L. S. PONTRYAGIN, *Continuous Groups*, Nauka Publ. Co. (Moscow, 1973). (Russian).

[22] W. TAYLOR, Varieties of topological algebras, *J. Austral. Math. Soc. Ser. A*, **23** (1977), 207—241.

[23] H. WERNER, A duality for weakly associative lattices, in: *Finite Algebra and Multiple-valued Logic*, Coll. Math. Soc. János Bolyai, vol. 28, North-Holland, Amsterdam, 1981; pp. 781—808.

(D.M.C.)
STATE UNIVERSITY COLLEGE
NEW PALTZ, N.Y.
U.S.A.

(P.H.K.)
GESAMTHOCHSCHULE KASSEL
KASSEL
FEDERAL REPUBLIC OF GERMANY

Acta Sci. Math., 47 (1984), 41—54

A characterization of quasi-varieties in equality-free languages

P. ECSEDI-TÓTH

1. The result

1.1. By a type t, we mean an ordered quintuple $t=\langle \mathcal{R}, \mathcal{F}, \mathcal{C}, t_{\mathcal{R}}, t_{\mathcal{F}}\rangle$ where $\mathcal{R}, \mathcal{F}, \mathcal{C}$ are pairwise disjoint sets, $\mathcal{C}$ is not empty, $t_{\mathcal{R}}: \mathcal{R}\to\omega$, $t_{\mathcal{F}}: \mathcal{F}\to\omega$. By a structure of type t, an ordered quadruplet $\langle A, \langle R_r\rangle_{r\in\mathcal{R}}, \langle F_f\rangle_{f\in\mathcal{F}}, \langle C_c\rangle_{c\in\mathcal{C}}\rangle$ is meant where A is a nonvoid set and $R_r\in\mathbf{P}(^{(t_{\mathcal{R}}(r))}A)$, $F_f\colon {}^{(t_{\mathcal{F}}(f))}A\to A$, $C_c\in A$, for every $r\in\mathcal{R}$, $f\in\mathcal{F}$, $c\in\mathcal{C}$. (For any set B, $\mathbf{P}(B)$ stands for the power set of B; and if $n\in\omega$, then ${}^nB$ denotes the n-th Cartesian power of B.) We shall use German capitals for denoting structures. If $\mathfrak{A}$ is a structure of type t, then the universe A, the relations R_r, the functions F_f and the constants C_c of $\mathfrak{A}$ will also be denoted by $|\mathfrak{A}|$, $R_r^{(\mathfrak{A})}$, $F_f^{(\mathfrak{A})}$, $C_c^{(\mathfrak{A})}$, respectively.

1.2. For $i\in\{0, 1\}$, let $\mathfrak{A}_i = \langle A^i, \langle R_r^i\rangle_{r\in\mathcal{R}}, \langle F_f^i\rangle_{f\in\mathcal{F}}, \langle C_c^i\rangle_{c\in\mathcal{C}}\rangle$ be two structures for a fixed type $t=\langle \mathcal{R}, \mathcal{F}, \mathcal{C}, t_{\mathcal{R}}, t_{\mathcal{F}}\rangle$. We define

$$\mathfrak{A}_0\cap\mathfrak{A}_1 = \langle A^0\cap A^1, \langle R_r^0\cap R_r^1\rangle_{r\in\mathcal{R}}, \langle F_f^0\cap F_f^1\rangle_{f\in\mathcal{F}}, \langle C_c^0\rangle_{c\in\mathcal{C}}\cap\langle C_c^1\rangle_{c\in\mathcal{C}}\rangle.$$

The *meet* of $\mathfrak{A}_0$ and $\mathfrak{A}_1$, in notation, $\mathfrak{A}_0 \mathbin{\pmb{\cap}} \mathfrak{A}_1$ is then defined as being identical to $\mathfrak{A}_0\cap\mathfrak{A}_1$ iff $\mathfrak{A}_0\cap\mathfrak{A}_1$ is itself a structure of type t.

Obviously, the meet of structures is a partial operation.

A class $\mathbf{K}$ of structures with the same type t is *closed under finite meets* iff for every $\mathfrak{A}_0, \mathfrak{A}_1\in\mathbf{K}$, if $\mathfrak{A}_0 \mathbin{\pmb{\cap}} \mathfrak{A}_1$ exists, then it is in $\mathbf{K}$.

1.3. Our result is the following preservation theorem: *If Σ is a set of equality-free sentences and $\mathbf{K}$ is the class of all models of Σ, then $\mathbf{K}$ is closed under finite meets iff Σ is equivalent to a set of universal equality-free Horn sentences* (Theorem 2.14).

Received December 7, 1981.

We note, that as a simple example will show, this theorem fails to hold for Σ with equality.

From this result we shall derive a characterization of quasi-varieties in equality-free languages. To be more specific, we shall prove, that *if* $\mathbf{K}$ *is an arbitrary class of structures for an equality-free language, then* $\mathbf{K}$ *is a quasi-variety* (i.e. $\mathbf{K}$ is axiomatizable by universal equality-free Horn sentences) *iff* $\mathbf{K}$ *is closed under finite meets, ultraproducts and equality-free elementary equivalence.*

1.4. Model theorists are generally uninterested in equality-free languages, since their expressive power is restricted in comparison with languages containing equality, and, on the other hand, several results and methods developed in the model theory for general first order languages apply directly to the equality-free case.

Recently, however, equality-free languages play a role in theoretical, as well as in practical computer science. For example, computability can be formalized in equality-free languages as was noticed by R. Hill and proved in general by H. ANDRÉKA and I. NÉMETI [1]. R. KOWALSKI [7] used this observation to show how equality-free languages can be considered as programming languages. On this basis, a practical programming language PROLOG was implemented by A. COLMERAUER et al [3]. A special case of our Lemma 2.13 was used by R. HILL [6] to prove a completeness theorem for a particularly efficient deduction system. Also, M. H. VAN EMDEN and R. KOWALSKI [4] investigated logically based programming languages by means of a special instance (for Herbrandian models) of the "easy direction" of the preservation theorem above, and questioned whether the converse was true. Our result shows that, generally, the answer is in the negative if equality is present in the language, but is affirmative if the equality is excluded. These applications make the belief plausible, that studying equality-free languages has some theoretical and practical value. This paper takes a step in this direction. Although motivations come from computer science, the preservation theorem mentioned above and its proof are purely model theoretic in character.

Acknowledgement. The author is indebted to E. Fried for his encouragement and to H. Andréka and I. Németi for their patience in correcting errors in an earlier version of the paper.

2. The proof

2.1. From now on, we shall fix an arbitrary type t. When we are speaking of an arbitrary structure without a closer indication of its type, then we shall always mean that it is a structure of the fixed type t.

2.2. Let $\mathfrak{A}=\langle|\mathfrak{A}|, \langle R_r^{(\mathfrak{A})}\rangle_{r\in\mathcal{R}}, \langle F_f^{(\mathfrak{A})}\rangle_{f\in\mathcal{F}}, \langle C_c^{(\mathfrak{A})}\rangle_{c\in\mathcal{C}}\rangle$ be an arbitrary structure. Let us suppose, that Δ is a congruence relation on $\mathfrak{A}$ (in the sense of [5, Definition

0.2.20, p. 73.]). We say, that Δ is a *universal congruence* relation on $\mathfrak{A}$, if

(1) $(\forall r \in \mathscr{R})(\forall a, b \in {}^{(t_{\mathscr{R}}(r))}|\mathfrak{A}|)\left[(\forall i < t_{\mathscr{R}}(r))\left(\langle a_i, b_i\rangle \in \Delta \rightarrow (a \in R_r^{(\mathfrak{A})} \leftrightarrow b \in R_r^{(\mathfrak{A})})\right)\right]$

where a_i, b_i denote the i-th component of a, b, respectively, for all $i < t_{\mathscr{R}}(r)$.

If Δ is a universal congruence on $\mathfrak{A}$, then the *quotient structure* $\mathfrak{A}/\Delta = \langle |\mathfrak{A}/\Delta|, \langle R_r^{(\mathfrak{A}/\Delta)}\rangle_{r\in\mathscr{R}}, \langle F_f^{(\mathfrak{A}/\Delta)}\rangle_{f\in\mathscr{F}}, \langle C_c^{(\mathfrak{A}/\Delta)}\rangle_{c\in\mathscr{C}}\rangle$ of $\mathfrak{A}$ *over* Δ can be defined in the traditional manner: for all $a \in |\mathfrak{A}|$, let $a/\Delta = \{b \mid b \in |\mathfrak{A}| \wedge \langle a, b\rangle \in \Delta\}$ and set

(2) $$|\mathfrak{A}/\Delta| = \{a/\Delta \mid a \in |\mathfrak{A}|\};$$

for all $f \in \mathscr{F}$, such that $t_{\mathscr{F}}(f) = n+1$ and for arbitrary $a_0, \ldots, a_n \in |\mathfrak{A}|$, we define

(3) $$F_f^{(\mathfrak{A}/\Delta)}(a_0/\Delta, \ldots, a_n/\Delta) = \left(F_f^{(\mathfrak{A})}(a_0, \ldots, a_n)\right)/\Delta;$$

for all $c \in \mathscr{C}$, let

(4) $$C_c^{(\mathfrak{A}/\Delta)} = C_c^{(\mathfrak{A})}/\Delta$$

and finally, for all $r \in \mathscr{R}$ such that $t_{\mathscr{R}}(r) = n+1$ and for every $a_0, \ldots, a_n \in |\mathfrak{A}|$, we define

(5) $$\langle a_0/\Delta, \ldots, a_n/\Delta\rangle \in R_r^{(\mathfrak{A}/\Delta)} \quad \text{iff} \quad \langle a_0, \ldots, a_n\rangle \in R_r^{(\mathfrak{A})}.$$

Δ being an universal congruence on $\mathfrak{A}$, the definition of $\mathfrak{A}/\Delta$ is correct and obviously, it is a structure of type t.

Proposition 2.3. *Let $\mathfrak{A}$ be an arbitrary structure and suppose, that Δ is a universal congruence on $\mathfrak{A}$. Then $\mathfrak{A} \equiv \mathfrak{A}/\Delta$, where elementary equivalence is meant in the equality-free sense.*

Proof. Let V be the set of variables and $k: V \rightarrow |\mathfrak{A}|$ be an arbitrary assignment relative to $\mathfrak{A}$. We define $\tilde{k}: V \rightarrow |\mathfrak{A}/\Delta|$ for all $v \in V$, by

(6) $$\tilde{k}(v) = k(v)/\Delta.$$

We shall prove by a straight-forward induction, that for arbitrary (equality-free) formula φ and $k: V \rightarrow |\mathfrak{A}|$,

(7) $$\mathfrak{A} \models \varphi[k] \quad \text{iff} \quad \mathfrak{A}/\Delta \models \varphi[\tilde{k}].$$

Since Δ is a congruence on $\mathfrak{A}$, it is easily seen, that for any term τ and $k: V \rightarrow |\mathfrak{A}|$,

(8) $$\tau^{(\mathfrak{A})}[k]/\Delta = \tau^{(\mathfrak{A}/\Delta)}[\tilde{k}].$$

(The notations used here are standard and can be found e.g. in [2, 1.3.1, 1.3.2, 1.3.3, pp. 22—23 and 1.3.13, 1.3.14, 1.3.15, pp. 27—28].)

Indeed, if $\tau \in V$ or $\tau \in \mathscr{C}$, then (6) or (4) is the same as (8). Let τ be of the form $f(\tau_0, \ldots, \tau_n)$ for some $f \in \mathscr{F}$, such that $t_{\mathscr{F}}(f) = n+1$ where τ_i is a term for which

(8) holds, for all $i<t_{\mathscr{F}}(f)$. Then,

$$\tau^{(\mathfrak{A})}[k]/\Delta=(F_f^{(\mathfrak{A})}(\tau_0^{(\mathfrak{A})}[k],\ldots,\tau_n^{(\mathfrak{A})}[k]))/\Delta \overset{(3)}{=} F_f^{(\mathfrak{A}/\Delta)}(\tau_0^{(\mathfrak{A})}[k]/\Delta,\ldots,\tau_n^{(\mathfrak{A})}[k]/\Delta) \overset{\text{(i.h.)}}{=}$$
$$\overset{\text{(i.h.)}}{=} F_f^{(\mathfrak{A}/\Delta)}(\tau_0^{(\mathfrak{A}/\Delta)}[\tilde{k}],\ldots,\tau_n^{(\mathfrak{A}/\Delta)}[\tilde{k}])=\tau^{(\mathfrak{A}/\Delta)}[\tilde{k}].$$

(Here, and everywhere below (i.h.) stands for "by the induction hypothesis".)

Turning to the proof of (7), we proceed similarly. Let φ be a prime formula of the form $r(\tau_0,\ldots,\tau_n)$, where $r\in\mathscr{R}, t_{\mathscr{R}}(r)=n+1$ and for all $i<t_{\mathscr{R}}(r)$, τ_i is a term. Then

$$\mathfrak{A}\models\varphi[k]\Leftrightarrow\mathfrak{A}\models r(\tau_0,\ldots,\tau_n)[k]\Leftrightarrow\langle\tau_0^{(\mathfrak{A})}[k],\ldots,\tau_n^{(\mathfrak{A})}[k]\rangle\in R_r^{(\mathfrak{A})}\overset{(5)}{\Leftrightarrow}$$
$$\overset{(5)}{\Leftrightarrow}\langle\tau_0^{(\mathfrak{A})}[k]/\Delta,\ldots,\tau_n^{(\mathfrak{A})}[k]/\Delta\rangle\in R_r^{(\mathfrak{A}/\Delta)}\overset{(8)}{\Leftrightarrow}\langle\tau_0^{(\mathfrak{A}/\Delta)}[\tilde{k}],\ldots,\tau_0^{(\mathfrak{A}/\Delta)}[\tilde{k}]\rangle\in R_r^{(\mathfrak{A}/\Delta)}\Leftrightarrow$$
$$\Leftrightarrow\mathfrak{A}/\Delta\models r(\tau_0,\ldots,\tau_n)[\tilde{k}]\Leftrightarrow\mathfrak{A}/\Delta\models\varphi[\tilde{k}].$$

If φ is of the form $\neg\psi$ and (7) is true for ψ, then

$$\mathfrak{A}\models\varphi[k]\Leftrightarrow\mathfrak{A}\models\neg\psi[k]\Leftrightarrow\mathfrak{A}\not\models\psi[k]\overset{\text{(i.h.)}}{\Leftrightarrow}\mathfrak{A}/\Delta\not\models\psi[\tilde{k}]\Leftrightarrow$$
$$\Leftrightarrow\mathfrak{A}/\Delta\models\neg\psi[\tilde{k}]\Leftrightarrow\mathfrak{A}/\Delta\models\varphi[\tilde{k}].$$

Obviously, the induction goes through for $\varphi=\psi_1\wedge\psi_2$.

Finally, let us suppose, that φ is of the form $\exists v\psi$, where $v\in V$ and (7) is true for ψ. Then,

(9) $\mathfrak{A}\models\varphi[k]\Leftrightarrow\mathfrak{A}\models\exists v\psi[k]\Leftrightarrow$ (there exists an assignment $k'\colon V\to|\mathfrak{A}|$ such that for all $w\in V, k'(w)=k(w)$, provided $v\neq w$ and $\mathfrak{A}\models\psi[k']$).

By the induction hypothesis, $\mathfrak{A}\models\psi[k'']$ iff $\mathfrak{A}/\Delta\models\psi[\tilde{k}'']$ for arbitrary $k''\colon V\to|\mathfrak{A}|$; moreover, for k and k' in (9), we have $k(w)=k'(w)$, for all $w\in V$ such that $v\neq w$. Thus (9) is equivalent to the assertion

(there exists an assignment $\tilde{k}'\colon V\to|\mathfrak{A}/\Delta|$ such that for all $w\in V$ if $v\neq w$ then $k(w)=k'(w)$ and $\mathfrak{A}/\Delta\models\psi[\tilde{k}']$),

which, in turn, is equivalent to $\mathfrak{A}/\Delta\models\exists v\psi[\tilde{k}]$.

This lemma has been proposed to me by H. Andréka and I. Németi to replace my original stronger but much less true assertion.

It is easy to construct a simple counterexample, using the obvious fact that an equation can hold in the quotient structure even if it is false in the initial one, which shows that Proposition 2.3 does not generalize for languages with equality.

2.4. Let X be an arbitrary set and consider the absolutely free algebra $\mathfrak{Fr}_{X\cup\mathscr{C}}$ of type t generated by the set $X\cup\mathscr{C}$ (cf. [5, Definition 0.4.19 (i), Remarks 0.4.20, pp. 130—131]).

Let $\mathfrak{A}$ be any structure of type t. It is well-known that, for arbitrary $h: X\cup\mathscr{C}\to|\mathfrak{A}|$ such that for all $c\in\mathscr{C}$, $h(c)=C_c^{(\mathfrak{A})}$ holds, there exists a unique homomorphism $\bar{h}$ from $\mathfrak{Fr}_{X\cup\mathscr{C}}$ into $\mathfrak{A}$ for which $h\subseteq\bar{h}$ (cf. Definition 0.4.23., Theorem 0.4.24, Theorem 0.4.27 (i), pp. 131—132, in [5]).

We define the *free structure* $\mathfrak{Fr}_h\mathfrak{A}$ induced by h and $\mathfrak{A}$ as follows. Let

(10) (i) $$|\mathfrak{Fr}_h\mathfrak{A}| = |\mathfrak{Fr}_{X\cup\mathscr{C}}|.$$

(ii) For every $r\in\mathscr{R}$, such that $t_{\mathscr{R}}(r)=n+1$ and for arbitrary $a_0, \ldots, a_n\in|\mathfrak{Fr}_h\mathfrak{A}|$, let

(11) $$\langle a_0, \ldots, a_n\rangle\in R_r^{(\mathfrak{Fr}_h\mathfrak{A})} \quad\text{iff}\quad \langle\bar{h}(a_0), \ldots, \bar{h}(a_n)\rangle\in R_r^{(\mathfrak{A})},$$

where $\bar{h}$ is the unique extension of h to a homomorphism from $\mathfrak{Fr}_{X\cup\mathscr{C}}$ into $\mathfrak{A}$.

(iii) For every $f\in\mathscr{F}$ such that $t_{\mathscr{F}}(f)=n+1$ and for arbitrary $a_0, \ldots, a_n\in|\mathfrak{Fr}_h\mathfrak{A}|$, let

(12) $$F_f^{(\mathfrak{Fr}_h\mathfrak{A})}(a_0, \ldots, a_n) = F_f^{(\mathfrak{Fr}_{X\cup\mathscr{C}})}(a_0, \ldots, a_n).$$

(iv) Finally, for all $c\in\mathscr{C}$, let

(13) $$C_c^{(\mathfrak{Fr}_h\mathfrak{A})} = C_c^{(\mathfrak{Fr}_{X\cup\mathscr{C}})}.$$

It is obvious, that the homomorphism $\bar{h}$ from $\mathfrak{Fr}_{X\cup\mathscr{C}}$ into $\mathfrak{A}$ is a homomorphism as well from $\mathfrak{Fr}_h\mathfrak{A}$ into $\mathfrak{A}$. Moreover, the relation

(14) $$\Delta_{\bar{h}} = \{\langle a, b\rangle | a, b\in|\mathfrak{Fr}_h\mathfrak{A}|\wedge\bar{h}(a) = \bar{h}(b)\}$$

is a universal congruence on $\mathfrak{Fr}_h\mathfrak{A}$. Thus, $\mathfrak{Fr}_h\mathfrak{A}$ is correctly defined and is a structure of type t. Additionally, $\mathfrak{Fr}_h\mathfrak{A}$ has the following useful property.

Lemma 2.5. *Let $\mathfrak{A}$ be an arbitrary structure, X be a set and $h: X\cup\mathscr{C}\to|\mathfrak{A}|$ be such that $h(c)=C_c^{(\mathfrak{A})}$ for all $c\in\mathscr{C}$. If h is onto, then $\mathfrak{Fr}_h\mathfrak{A}\equiv\mathfrak{A}$ where elementary equivalence is meant in the equality-free sense.*

Proof. Since $\Delta_{\bar{h}}$, defined by (14) is a universal congruence on $\mathfrak{Fr}_h\mathfrak{A}$, we have from Proposition 2.3 that $\mathfrak{Fr}_h\mathfrak{A}\equiv\mathfrak{Fr}_h\mathfrak{A}/\Delta_{\bar{h}}$. On the other hand, $\mathfrak{Fr}_h\mathfrak{A}/\Delta_{\bar{h}}\cong\mathfrak{A}$ by the isomorphism g defined as $g(a/\Delta_{\bar{h}})=\bar{h}(a)$. Hence, $\mathfrak{Fr}_h\mathfrak{A}/\Delta_{\bar{h}}\equiv\mathfrak{A}$, which yields to the assertion.

An important consequence of this lemma is formulated as follows.

Corollary 2.6. *Let $\mathbf{K}\neq\emptyset$ be any class of structures which is closed under equality-free elementary equivalence. Let $\mathfrak{A}, \mathfrak{B}\in\mathbf{K}$. Then, there exist $\mathfrak{A}', \mathfrak{B}'\in\mathbf{K}$ such that the following conditions are satisfied: $\mathfrak{A}\equiv\mathfrak{A}'$, $\mathfrak{B}\equiv\mathfrak{B}'$ and the meet $\mathfrak{A}'\sqcap\mathfrak{B}'$ exists. (Elementary equivalence is meant in the equality-free sense.)*

Proof. Let X be a set with cardinality large enough such that $\operatorname{card}(X\cup\mathscr{C})\geqq\operatorname{card}(|\mathfrak{A}|\cup|\mathfrak{B}|)$. Define $h_1\colon X\cup\mathscr{C}\to|\mathfrak{A}|$ and $h_2\colon X\cup\mathscr{C}\to|\mathfrak{B}|$ in such a way that both h_1 and h_2 are onto and for all $c\in\mathscr{C}$, $h_1(c)=C_c^{(\mathfrak{A})}$ and $h_2(c)=C_c^{(\mathfrak{B})}$. Let $\mathfrak{A}'=\mathfrak{Fr}_{h_1}\mathfrak{A}$ and $\mathfrak{B}'=\mathfrak{Fr}_{h_2}\mathfrak{B}$. By Lemma 2.5., $\mathfrak{A}\equiv\mathfrak{A}'$ and $\mathfrak{B}\equiv\mathfrak{B}'$. Since $\mathbf{K}$ is closed under equality-free elementary equivalence, we have $\mathfrak{A}'\in\mathbf{K}$ and $\mathfrak{B}'\in\mathbf{K}$. It remains to establish that $\mathfrak{A}'\sqcap\mathfrak{B}'$ exists.

Obviously,

$$|\mathfrak{A}'\cap\mathfrak{B}'|=|\mathfrak{Fr}_{h_1}\mathfrak{A}|\cap|\mathfrak{Fr}_{h_2}\mathfrak{B}|\overset{(10)}{=}|\mathfrak{Fr}_{X\cup\mathscr{C}}|\cap|\mathfrak{Fr}_{X\cup\mathscr{C}}|=|\mathfrak{Fr}_{X\cup\mathscr{C}}|\supseteqq X\cup\mathscr{C},$$

hence, the universe of $\mathfrak{A}'\sqcap\mathfrak{B}'$ is not empty. Similarly, for all $f\in\mathscr{F}$,

$$F_f^{(\mathfrak{A}')}\cap F_f^{(\mathfrak{B}')}=F_f^{(\mathfrak{Fr}_{h_1}\mathfrak{A})}\cap F_f^{(\mathfrak{Fr}_{h_2}\mathfrak{B})}\overset{(12)}{=}F_f^{(\mathfrak{Fr}_{X\cup\mathscr{C}})}\cap F_f^{(\mathfrak{Fr}_{X\cup\mathscr{C}})}=F_f^{(\mathfrak{Fr}_{X\cup\mathscr{C}})},$$

hence the set theoretic meet of the functions rendered to f in $\mathfrak{A}'$ and $\mathfrak{B}'$, respectively, is again a function.

For all $c\in\mathscr{C}$,

$$C_c^{(\mathfrak{A}')}=C_c^{(\mathfrak{Fr}_{h_1}\mathfrak{A})}\overset{(13)}{=}C_c^{(\mathfrak{Fr}_{X\cup\mathscr{C}})}\overset{(13)}{=}C_c^{(\mathfrak{Fr}_{h_2}\mathfrak{B})}=C_c^{(\mathfrak{B}')},$$

thus the meet of the two sequences $\langle C_c^{(\mathfrak{A}')}\rangle_{c\in\mathscr{C}}$ and $\langle C_c^{(\mathfrak{B}')}\rangle_{c\in\mathscr{C}}$ is the sequence $\langle C_c^{(\mathfrak{Fr}_{X\cup\mathscr{C}})}\rangle_{c\in\mathscr{C}}$.

Finally, for any $r\in\mathscr{R}$, the meet of the two relations $R_r^{(\mathfrak{A}')}$ and $R_r^{(\mathfrak{B}')}$ is again a relation with arity $t_{\mathscr{R}}(r)$.

Hence $\mathfrak{A}'\cap\mathfrak{B}'$ is a structure of type t, whence $\mathfrak{A}'\sqcap\mathfrak{B}'$ exists.

This corollary shows that the property "a class $\mathbf{K}$ of structures is closed under finite meets" is not trivial in the case when $\mathbf{K}$ is the class of all models of a set of sentences.

2.7. We shall need the well-known preservation theorem under submodels (cf. [2], Theorem 3.2.2, p. 124, or Theorem 5.2.4, p. 228) in a somewhat stronger form: i.e. for equality-free languages. (Insufficiency of the original form for our purposes in the proof of Lemma 2.11, below, was pointed out to me by H. Andréka and I. Németi.) Before formulating the stronger version some preparation is required.

Let $\mathfrak{A}$ be an arbitrary structure and let Γ be the diagram of $\mathfrak{A}$ (cf. [2], p. 68). Let

$$\Gamma_{\mathfrak{A}}=\{\varphi\mid\varphi\in\Gamma\text{ and }\varphi\text{ does not contain the equality symbol}\}.$$

The set $\Gamma_{\mathfrak{A}}$ will be called the *equality-free diagram* of $\mathfrak{A}$.

Lemma 2.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures. If the equality-free diagram $\Gamma_{\mathfrak{A}}$ holds in $\mathfrak{B}$, then there exist two structures $\mathfrak{A}'$ and $\mathfrak{B}'$ of the diagram types* *of* $\mathfrak{A}$ *and* $\mathfrak{B}$, *respectively, such that the following conditions hold:* $\mathfrak{A}\subseteq\mathfrak{A}'$, $\mathfrak{B}'\subseteq\mathfrak{B}$ *and* $\mathfrak{A}'\equiv\mathfrak{B}'$ *in the equality-free sense.*

Proof. For all $a\in|\mathfrak{A}|$, let c_a be a completely new constant symbol. Let $\mathfrak{A}'=(\mathfrak{A}, a)_{a\in|\mathfrak{A}|}$ and let $\mathfrak{B}'$ be that submodel of $\mathfrak{B}$ which is generated by the set $\{C_{c_a}^{(\mathfrak{B})}\mid a\in|\mathfrak{A}|\}$ where c_a is the new constant symbol for $a\in|\mathfrak{A}|$. Clearly, $\mathfrak{A}\subseteq\mathfrak{A}'$ and $\mathfrak{B}'\subseteq\mathfrak{B}$; moreover it follows that $\mathfrak{B}'\models\Gamma_{\mathfrak{A}}$.

Let φ be an equality-free sentence. We prove by induction on the construction of φ, that

(15) $$\mathfrak{A}'\models\varphi \quad\text{iff}\quad \mathfrak{B}'\models\varphi.$$

Let φ be a prime sentence of the form $r(\tau_0, \ldots, \tau_n)$, where $\tau_0, \ldots, \tau_n$ are terms in which no variables occur, $r\in\mathscr{R}$, $t_{\mathscr{R}}(r)=n+1$. If $\mathfrak{A}'\models r(\tau_0, \ldots, \tau_n)$, then $r(\tau_0, \ldots, \tau_n)\in\Gamma_{\mathfrak{A}}$, thus $\mathfrak{B}'\models r(\tau_0, \ldots, \tau_n)$. Similarly, if $\mathfrak{A}'\not\models r(\tau_0, \ldots, \tau_n)$, then $\neg r(\tau_0, \ldots, \tau_n)\in\Gamma_{\mathfrak{A}}$ and so $\mathfrak{B}'\models\neg r(\tau_0, \ldots, \tau_n)$, i.e. $\mathfrak{B}'\not\models r(\tau_0, \ldots, \tau_n)$.

It is obvious, that the induction goes through for the cases $\neg\psi$ and $\psi_1\wedge\psi_2$.

Let φ be a sentence of the form $\exists v\psi$. If $\mathfrak{A}'\models\varphi$, then there exists $k\colon V\to|\mathfrak{A}'|$ such that $\mathfrak{A}'\models\psi[k]$. It follows, that the sentence ψ^*, obtained from ψ by substituting every occurrence of v by $c_{k(v)}$, holds in $\mathfrak{A}'$. By the induction hypothesis, (15) is true for ψ^*, hence $\mathfrak{B}'\models\psi^*$. Let $k'\colon V\to|\mathfrak{B}'|$ be such that $k'(v)=C_{c_{k(v)}}^{(\mathfrak{B}')}$ (and arbitrary otherwise). Then, $\mathfrak{B}'\models\psi[k']$. Hence, there is a $k'\colon V\to|\mathfrak{B}'|$ for which $\mathfrak{B}'\models\psi[k']$, whence $\mathfrak{B}'\models\exists v\psi$.

Conversely, let us suppose, that $\mathfrak{B}'\models\exists v\psi$. Then there exists an assignment $k'\colon V\to|\mathfrak{B}'|$, such that $\mathfrak{B}'\models\psi[k']$. By the definition of $\mathfrak{B}'$, there exists a term τ, in which no variables occur, such that $k'(v)=\tau^{(\mathfrak{B}')}$ and so, the sentence ψ^*, obtained again from ψ by substituting v everywhere by the term τ, holds in $\mathfrak{B}'$. By the induction hypothesis, we have $\mathfrak{A}'\models\psi^*$. But then, there exists a $k\colon V\to|\mathfrak{A}'|$ such that $k(v)=\tau^{(\mathfrak{A}')}$ and thus $\mathfrak{A}'\models\psi[k]$. So, there is an assignment $k\colon V\to|\mathfrak{A}'|$ for which $\mathfrak{A}'\models\psi[k]$, whence $\mathfrak{A}'\models\exists v\psi$.

We quote Lemma 3.2.1 from [2], p. 124.

Lemma 2.9. *Let* Σ *be a consistent set of sentences in an arbitrary first order language* L *and let* Γ *be a set of sentences in* L *which is closed under finite disjunctions. Then, the following two assertions are equivalent:*

(i) Σ *has a set of axioms* Σ_1 *such that* $\Sigma_1\subseteq\Gamma$.

(ii) *If* $\mathfrak{A}$ *is a model of* Σ *and every sentence* $\varphi\in\Gamma$ *holding in* $\mathfrak{A}$ *holds in* $\mathfrak{B}$, *then* $\mathfrak{B}$ *is a model of* Σ.

It is easy to check that the proof of this lemma in [2], p. 124, does not depend on the presence or lack of equality, hence we can use it for equality-free languages, as well.

The next assertion is the stronger form of the preservation theorem concerning submodels.

Lemma 2.10. *Let Σ be a set of sentences in an arbitrary equality-free language L. Then, the following two conditions are equivalent:*

(i) *Σ is preserved under submodels,*

(ii) *Σ has a set of equality-free universal axioms.*

Proof. It is immediate that (ii) entails (i.) To prove the converse, let us suppose that Σ is preserved under submodels, and let $\mathfrak{A}$ be a model of Σ. Let $\mathfrak{B}$ be such that every equality-free universal sentence holding in $\mathfrak{A}$ holds in $\mathfrak{B}$. Then every existential equality-free sentence true in $\mathfrak{B}$ is true in $\mathfrak{A}$. For if this is not the case, i.e. there exists an existential equality-free sentence, say φ, such that $\mathfrak{B}\models\varphi$ and $\mathfrak{A}\not\models\varphi$, then $\mathfrak{A}\models\neg\varphi$. But $\neg\varphi$ is a universal equality-free sentence, thus, by assumption $\mathfrak{B}\models\neg\varphi$.

Consider the theory $\Sigma'=\Sigma\cup\Gamma_{\mathfrak{B}}$, where $\Gamma_{\mathfrak{B}}$ is the equality-free diagram of $\mathfrak{B}$. Σ' is consistent (provided Σ is such), since for any finite set

$$\{\theta_0(b_0, \dots, b_n), \dots, \theta_m(b_0, \dots, b_n)\}\subseteqq\Gamma_{\mathfrak{B}},$$

the existential equality-free sentence

$$\psi = (\exists x_0 \dots \exists x_n)\big(\theta_0(x_0, \dots, x_n)\wedge \dots \wedge\theta_m(x_0, \dots, x_n)\big)$$

is true in $\mathfrak{B}$, hence in $\mathfrak{A}$, too. Thus, ψ is consistent with Σ. By compactness; Σ' is consistent, and has a model $\mathfrak{C}$. So we have $\mathfrak{C}\models\Sigma$ and $\mathfrak{C}\models\Gamma_{\mathfrak{B}}$. By Lemma 2.8 there exist structures $\mathfrak{B}', \mathfrak{C}'$ such that $\mathfrak{C}\supseteqq\mathfrak{C}'$, $\mathfrak{B}\subseteqq\mathfrak{B}'$ and $\mathfrak{B}'\equiv\mathfrak{C}'$ in the equality-free sense. Σ is preserved under submodels, thus $\mathfrak{C}'\models\Sigma$ and so $\mathfrak{B}'\models\Sigma$. Again, by the preservation property of Σ, $\mathfrak{B}\models\Sigma$.

Let Γ be the set of all sentences, which are equivalent to universal equality-free sentences. Obviously, Γ is closed under finite disjunctions. Thus, the conditions of Lemma 2.9 (ii) are satisfied, and we obtain from Lemma 2.9 (i), that Σ has a set of universal equality-free axioms.

This proof follows closely the proof of Theorem 3.2.2 in [2], p. 128. The only difference, that we use the equality-free diagram of $\mathfrak{B}$ in place of the diagram of $\mathfrak{B}$ in the original proof.

Lemma 2.11. *Let L be an equality-free language and Σ be a set of sentences in L. If Σ is preserved under finite meets, then Σ has a set of universal axioms in L.*

Proof. Let us suppose, that Σ is preserved under finite meets, i.e. if $\mathfrak{A}\models\Sigma$ and $\mathfrak{B}\models\Sigma$, then $\mathfrak{A}\sqcap\mathfrak{B}\models\Sigma$, provided the meet exists. In contrary to the assertion, let us assume, that no set of universal equality-free axioms exists for Σ. Then Σ is consistent and by Lemma 2.10, there exist $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{B}\subseteq\mathfrak{A}$, $\mathfrak{A}\models\Sigma$, $\mathfrak{B}\not\models\Sigma$ hold. Let us define $\mathfrak{A}'$ as follows. We set first

$$|\mathfrak{A}'| = |\mathfrak{B}|\cup((|\mathfrak{A}|-|\mathfrak{B}|)\times\{|\mathfrak{A}|\}).$$

Let $h\colon |\mathfrak{A}|\to|\mathfrak{A}'|$ be a mapping such that

(16) $$h(b)=b \text{ for all } b\in|\mathfrak{B}|,$$

(17) $$h(a)=\langle a, |\mathfrak{A}|\rangle \text{ for all } a\in|\mathfrak{A}|-|\mathfrak{B}|.$$

Clearly, h is one-one and onto, hence we can define:

(18) $$R_r^{(\mathfrak{A}')} = \{\langle a_0, \ldots, a_n\rangle \mid a_0, \ldots, a_n\in|\mathfrak{A}'|\wedge\langle h^{-1}(a_0), \ldots, h^{-1}(a_n)\rangle\in R_r^{(\mathfrak{A})}\}$$

for all $r\in\mathscr{R}$, such that $t_{\mathscr{R}}(r)=n+1$;

(19) $$F_f^{(\mathfrak{A}')}(a_0, \ldots, a_n) = h\big(F_f^{(\mathfrak{A})}(h^{-1}(a_0), \ldots, h^{-1}(a_n))\big)$$

for all $f\in\mathscr{F}$, $t_{\mathscr{F}}(f)=n+1$ and $a_0, \ldots, a_n\in|\mathfrak{A}'|$;

(20) $$C_c^{(\mathfrak{A}')} = h(C_c^{(\mathfrak{A})}).$$

It follows from (16), (17), (18), (19), (20), that $\mathfrak{A}'$ is isomorphic to $\mathfrak{A}$ (by h) and so $\mathfrak{A}'\equiv\mathfrak{A}$. Thus, we have $\mathfrak{A}'\models\Sigma$. The meet of $\mathfrak{A}$ and $\mathfrak{A}'$ exists by the construction and $\mathfrak{A}\sqcap\mathfrak{A}'=\mathfrak{B}$.

We obtain that Σ is not preserved under finite meets, a contradiction.

Lemma 2.12. *Let L be an equality-free language and Σ be a set of sentences in L. If Σ is preserved under finite meets, then Σ has a set of axioms consisting of universal equality-free Horn sentences.*

Proof. Let us suppose that Σ is preserved under finite meets, but, in contrary to the assertion, Σ has no set of axioms consisting of universal equality-free Horn sentences. By Lemma 2.11, however, Σ has a set Γ of universal equality-free axioms. It follows from the indirect assumption that Σ, hence Γ is consistent. Again, by the absurd hypothesis, there exists (at least one) sentence $\varphi\in\Gamma$ such that φ is equivalent to a sentence of the form

(21) $$(\forall x_0 \ldots \forall x_n) \bigwedge_{u=1}^{z} \left(\bigwedge_{i=1}^{s_u} p_{iu} \to \bigvee_{j=1}^{m_u} q_{ju}\right)$$

where $z, n\in\omega$ and for all u $(1\leqq u\leqq z)$, $s_u, m_u\in\omega$; moreover each p_{iu}, q_{ju} $(1\leqq i\leqq s_u$, $1\leqq j\leqq m_u)$ is a prime formula in which at most $x_0, \ldots, x_n$ can occur; and φ is not equivalent to any sentence of the form

$$(22) \qquad (\forall x_0 \dots \forall x_n) \bigwedge_{u=1}^{z} \left(\bigwedge_{i=1}^{s_u} p_{iu} \to q_{ju} \right)$$

where $n, z, s_u, p_{iu}, q_{ju}$ are just as in (21) and $j=1, \dots, m_u$. For the sake of simplicity, let us suppose that $z=1$ and $m_1=2$. Let $\mathfrak{A}$ be a structure such that $\mathfrak{A} \models \varphi$, but

$$\mathfrak{A} \not\models (\forall x_0 \dots \forall x_n) \left(\bigwedge_{i=1}^{s} p_i \to q_1 \right) \quad \text{and} \quad \mathfrak{A} \not\models (\forall x_0 \dots \forall x_n) \left(\bigwedge_{i=1}^{s} p_i \to q_2 \right).$$

(Such a structure $\mathfrak{A}$ exists by (21) and (22), and by the fact that Γ is consistent.) Then, for some $k_1 \colon V \to |\mathfrak{A}|$ and $k_2 \colon V \to |\mathfrak{A}|$, we have

$$\mathfrak{A} \not\models \left(\bigwedge_{i=1}^{s} p_i \to q_1 \right) [k_1] \quad \text{and} \quad \mathfrak{A} \not\models \left(\bigwedge_{i=1}^{s} p_i \to q_2 \right) [k_2].$$

Let B be a completely new set of constant symbols such that $\operatorname{card}(B \cup \mathscr{C}) \geqq \operatorname{card}(|\mathfrak{A}|)$. Let $b_0, \dots, b_n \in B$ be $n+1$ distinct elements. We can define the mappings $h_1 \colon B \cup \mathscr{C} \to |\mathfrak{A}|$ and $h_2 \colon B \cup \mathscr{C} \to |\mathfrak{A}|$ in such a way that both h_1 and h_2 are onto and the following conditions hold: $h_1(c) = h_2(c) = C_c^{(\mathfrak{A})}$ for all $c \in \mathscr{C}$, and $h_1(b_l) = k_1(x_l)$, $h_2(b_l) = k_2(x_l)$ for all $l \in \{0, 1, \dots, n\}$. Let $\mathfrak{A}_1 = \mathfrak{Fr}_{h_1} \mathfrak{A}$ and $\mathfrak{A}_2 = \mathfrak{Fr}_{h_2} \mathfrak{A}$. By Lemma 2.5, $\mathfrak{A}_1 \equiv \mathfrak{A} \equiv \mathfrak{A}_2$ in the equality-free sense and thus

$$\mathfrak{A}_1 \models \varphi, \quad \mathfrak{A}_2 \models \varphi,$$

$$(23) \qquad \mathfrak{A}_1 \not\models \left(\bigwedge_{i=1}^{s} p_i \to q_1 \right) [k_1'], \quad \mathfrak{A}_2 \not\models \left(\bigwedge_{i=1}^{s} p_i \to q_2 \right) [k_2'[$$

where $k_1' \colon V \to |\mathfrak{A}_1|$, $k_2' \colon V \to |\mathfrak{A}_2|$ such that $k_1'(x_l) = b_l = k_2'(x_l)$ for all $l \in \{0, 1, \dots, n\}$ (and arbitrary otherwise). Notice, that $|\mathfrak{A}_1| = |\mathfrak{A}_2|$ and $\{b_0, \dots, b_n\} \subseteq |\mathfrak{A}_1|$, hence the definitions of k_1' and k_2' are correct. We may assume that $k_1' = k_2'$. Moreover, using an analogous argument to the proof of Corollary 2.6, it is easily seen that $\mathfrak{A}_1 \sqcap \mathfrak{A}_2$ exists.

It follows from (23) and from $k_1' = k_2'$ that

$$\mathfrak{A}_1 \sqcap \mathfrak{A}_2 \not\models \left(\bigwedge_{i=1}^{s} p_i \to q_1 \vee q_2 \right) [k_1'],$$

hence $\mathfrak{A}_1 \sqcap \mathfrak{A}_2 \not\models \varphi$, contradicting the assumption that Σ is preserved under finite meets.

Using a simple induction, one obtains contradictions in a similar way for all $z \geqq 1$ and $m_u \geqq 2$.

We remark that this lemma is false if equality is present in the language. For example, let

$$\Sigma = \{(\forall x \forall y \forall z)(x \equiv y \vee x \equiv z)\}.$$

Clearly, Σ is consistent and is preserved under finite meets, for any structure $\mathfrak{A}$ is a model of Σ iff card $|\mathfrak{A}| \leqq 2$.

The following assertion is the converse of the lemma above, and establishes the easy direction of our preservation theorem. It is true, however, for arbitrary first order languages with equality.

Lemma 2.13. *Let L be an arbitrary first order language and Σ be a consistent set of sentences. If Σ has a set of axioms consisting of universal Horn sentences, then Σ is preserved under finite meets.*

Proof. It suffices to prove that every universal Horn sentence φ is preserved under finite meets. We proceed by induction on the construction of φ.

Let φ be a quantifier-free basic Horn formula and assume that the free variables of φ are among $\{x_0, \ldots, x_m\}$. By definition, φ is equivalent to one of the following two forms:

$$(\forall x_0 \ldots \forall x_m)\left(\bigwedge_{i=1}^{s} p_i \to q\right), \tag{24}$$

or

$$(\forall x_0 \ldots \forall x_m)\left(\neg \bigwedge_{i=1}^{s} p_i\right), \tag{25}$$

where $p_i\ (1 \leqq i \leqq s)$ and q are prime formulae and $s \in \omega$. (As usual, we allow $s=0$, in which case φ is equivalent either to q or is inconsistent; the latter possibility is, however, ruled out by the conditions.)

For illustration, we consider the case when φ is equivalent to a sentence of the form (24); the other one can be treated similarly.

Let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be two structures such that $\mathfrak{A}_0 \sqcap \mathfrak{A}_1$ exists and both structures are models of φ, i.e.

$$\mathfrak{A}_j \models (\forall x_0 \ldots \forall x_m)\left(\bigwedge_{i=1}^{s} p_i \to q\right) \quad (j = 0, 1). \tag{26}$$

Let $k\colon V \to |\mathfrak{A}_0 \sqcap \mathfrak{A}_1|$ be arbitrary and distinguish the following two subcases:

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models q[k], \tag{27}$$

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \not\models q[k]. \tag{28}$$

If (27) holds, then by propositional logic, we have immediately that

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models \left(\bigwedge_{i=1}^{s} p_i \to q\right)[k]. \tag{29}$$

If (28) is the case, then by the definition of the meet, it follows that for some $j\in\{0, 1\}$, $\mathfrak{A}_j\not\models q[k']$ where $k'\colon V\to|\mathfrak{A}_j|$ is such that $k'(x_l)=k(x_l)$ for all $l\in\{0, 1, \ldots, m\}$. From (26), we have that

$$\mathfrak{A}_j \not\models \bigwedge_{i=1}^{s} p_i[k'].$$

Again, by the definition of the meet, it follows, that

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \not\models \bigwedge_{i=1}^{s} p_i[k],$$

thus, by propositional logic:

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models \left(\bigwedge_{i=1}^{s} p_i \to q\right)[k]. \tag{30}$$

Putting (29) and (30) together, we obtain that for arbitrary k, (26) entails that

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models \left(\bigwedge_{i=1}^{s} p_i \to q\right)[k],$$

which is equivalent to

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models (\forall x_0 \ldots \forall x_m)\left(\bigwedge_{i=1}^{s} p_i \to q\right),$$

hence $\mathfrak{A}_0 \sqcap \mathfrak{A}_1$ is a model of φ.

It is clear, that the induction goes through for conjunctions of quantifier-free basic Horn formulae.

Let us suppose now, that φ is equivalent to a sentence of the form $(\forall x)\psi$, where ψ is a conjunction of quantifier-free basic Horn formulae. Assume that $\mathfrak{A}_0\models(\forall x)\psi$, $\mathfrak{A}_1\models(\forall x)\psi$ and that $\mathfrak{A}_0 \sqcap \mathfrak{A}_1$ exists.

Then, for arbitrary $k_0\colon V\to|\mathfrak{A}_0|$ and $k_1\colon V\to|\mathfrak{A}_1|$,

$$\mathfrak{A}_0 \models \psi[k_0] \quad\text{and}\quad \mathfrak{A}_1 \models \psi[k_1],$$

respectively. It follows that for arbitrary $k\colon V\to|\mathfrak{A}_0 \sqcap \mathfrak{A}_1|$, we have

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models \psi[k]$$

and thus $\mathfrak{A}_0 \sqcap \mathfrak{A}_1\models(\forall x)\psi$.

Applying a trivial induction on the number of universal quantifiers (in the prenex form of φ), the lemma is established.

We note that the proof does not depend on the form of prime formulae occurring in φ, hence the assertion is true for languages with equality, too.

Theorem 2.14. *Let L be an equality-free first order language; let Σ be a set of sentences in L and assume that $\mathbf{K}$ is the class of models for Σ. Then, the following two assertions are equivalent:*

(i) $\mathbf{K}$ *is closed under finite meets.*

(ii) Σ *is equivalent to a set of universal equality-free Horn sentences (in L).*

Proof. (i) entails (ii), by Lemma 2.12. The converse follows from Lemma 2.13, provided Σ is consistent. If Σ is inconsistent, then (i) trivially holds by definition.

We note that this theorem does not generalize to arbitrary first order languages. More precisely, (i) does not entail (ii) if equality is present in the language, as was shown by the counterexample after Lemma 2.12. The converse implication (ii)$\Rightarrow$(i) is, however, true in general.

2.16. Let L be an arbitrary equality-free language and $\mathbf{K}$ be a class of structures for L. $\mathbf{K}$ is an *elementary class* in L iff there exists a set Σ of sentences in L such that

$$\mathbf{K} = \{\mathfrak{A} \mid \mathfrak{A} \models \Sigma\}.$$

If Σ consists of universal equality-free Horn sentences, then $\mathbf{K}$ is said to be a *quasi-variety in L.*

The following assertion is a version of the well-known theorem of Łoś (cf. [2], Theorem 4.1.12, p. 173) for equality-free languages.

Lemma 2.17. *Let L be an arbitrary equality-free first order language and $\mathbf{K}$ be a class of structures for L. Then, $\mathbf{K}$ is an elementary class in L iff $\mathbf{K}$ is closed under equality-free elementary equivalence and ultraproducts.*

Proof. Completely the same as the proof of Theorem 4.1.12 in [2], p. 173.

Corollary 2.18. *Let L be an arbitrary equality-free first order language and $\mathbf{K}$ be a class of structures for L. Then the following two assertions are equivalent.*

(i) $\mathbf{K}$ *is closed under finite meets, ultraproducts and equality-free elementary equivalence.*

(ii) $\mathbf{K}$ *is a quasi-variety in L.*

Proof. Immediate by Theorem 2.15 and Lemma 2.17.

We note that this corollary does not generalize for languages with equality; more precisely (i) does not imply (ii) if the equality is present (cf. the counterexample after Lemma 2.12). The converse, however, holds for arbitrary first order languages, by Lemma 2.13 and Lemma 2.17.

References

[1] H. ANDRÉKA, I. NÉMETI, The generalized completeness of Horn predicate logic as a programming language, *Acta Cybernetica*, 4 (1978), 3—10.

[2] C. C. CHANG, H. J. KEISLER, *Model Theory*, North-Holland Publ. Co. (Amsterdam—London, 1973).

[3] A. COLMERAUER, H. KANOUI, R. PASERO, P. ROUSSELL, Un système de communication homme-machine en français, Rapport préliminaire, Groupe de Recherche en AI, U.E.R. de Luminy, Marseille, 1972.

[4] M. M. VAN EMDEN, R. A. KOWALSKI, The semantics of Predicate Logic as a programming language, *J. Assoc. Comput. Mach.*, 23 (1976), 733—742.

[5] L. HENKIN, J. D. MONK, A. TARSKI, *Cylindric Algebras*, Part I, North-Holland Publ. Co. (Amsterdam—London, 1971).

[6] R. HILL, LUSH-Resolution and Its Completeness, DCL Memo, No. 78., Univ. of Edinburgh, 1974.

[7] R. A. KOWALSKI, Predicate Logic as Programming Language in: *Proc. of IFIP'74*, North-Holland Publ. Co. (Amsterdam—London, 1974), pp. 569—574.

SOMOGYI B. U. 7.
6720 SZEGED, HUNGARY

Acta Sci. Math., 47 (1984), 55—59

Characterizations of some classes of semigroups

GYÖRGY POLLÁK and OTTÓ STEINFELD

Dedicated to E. S. Lyapin on his 70th birthday

In his paper [1], L. RÉDEI determined the structure of all rings A such that A contains divisors of zero but its proper subrings do not. In [2], O. STEINFELD proved that these rings coincide with those having the property that 0 is not a prime ideal in A but it is prime in every proper subring of A. R. WIEGANDT [4] gave some new characterizations of the same class, and in [5] he determined the structure of a larger class of rings. In [3], F. SZÁSZ proved the mentioned result of Steinfeld by a quite elementary method.

The purpose of the present note is to prove some theorems which can be considered as semigroup theoretical analogues of the structure theorems mentioned above. It turns out that Rédei's and Steinfeld's results as well as part of Wiegandt's equivalent conditions do have their exact counterparts for semigroups; however, as soon as one considers conditions involving left ideals, they fail to be equivalent to the former ones, and determine larger classes. Note that, to a certain extent, this was the case already for rings: the condition "R is non-cancellative but every proper left ideal of R is" was equivalent to the rest under the assumption of the descending chain condition [4] (for semigroups, the equivalence fails even in this case).

In Theorem 1, the equivalence $(\mathrm{i}_1)\Leftrightarrow(\mathrm{iii}_1)$ is the analogue of Rédei's result, $(\mathrm{i}_1)\Leftrightarrow(\mathrm{ii}_1)$ is that of Wiegandt's ([4], Theorem 1), and $(\mathrm{i}_1)\Leftrightarrow(\mathrm{iv}_1)$ corresponds to Steinfeld's theorem.

Theorem 1. *The following conditions are equivalent for a semigroup S with* 0:

(i_1) *S is either the* 0*-direct union of two groups of prime order with* 0 *or a two-element zero semigroup;*

(ii_1) *S is not a group with* 0, *but every proper subsemigroup of S is either a subgroup or a subgroup with* 0 *of S;*

Received October 11, 1982.

(iii$_1$) *S contains divisors of* 0, *but every proper subsemigroup of S is free from divisors of* 0;

(iv$_1$) (0) *is not a prime ideal of S but it is prime in every proper subsemigroup with* 0 *of S.*

Remark 1. Note that by a subgroup with 0 in (ii$_1$) we mean a subsemigroup having 0 for zero element; else the assertion were false (cf. the example in Remark 2).

Proof. (i$_1$)$\Rightarrow$(ii$_1$) is obvious.

(ii$_1$)$\Rightarrow$(iii$_1$). If S did not contain zero divisors, then $S\setminus\{0\}$ were a proper subsemigroup of S and therefore a group, which contradicts the assumption. The second assertion is obvious.

(iii$_1$)$\Rightarrow$(iv$_1$). If a and b are non-zero elements of S such that $ab=0$, then $(a\cup Sa)(b\cup bS)=0$. Put $Q=(a\cup Sa)\cap(b\cup bS)$, then $(a\cup Sa)Q=0$. If $Q=S$ then $S^2=Q^2\subseteqq(a\cup Sa)Q=0$ whence our first assertion holds true (the second one is trivial anyway). If $Q\neq S$ then $Q^2=0$ and the second condition in (iii$_1$) implies $Q=0$. Hence

$$(b\cup bS)(a\cup Sa)\subseteqq Q=0.$$

This means that (0) is not a prime ideal in S, in fact, it is easily seen that $(b)(a)=0$.

(iv$_1$)$\Rightarrow$(i$_1$). Let $0\neq J_1, J_2\lhd S, J_1J_2=0$. If J_1 had a non-trivial subsemigroup A (i.e. $A\neq J_1$, $A\neq 0$), then $A\cup J_2$ were a proper subsemigroup of S such that (0) is not prime in $A\cup J_2$, as $J_1\cap J_2=0$ follows from the second part of (iv$_1$). Hence J_1 is either a two-element zero semigroup or a group of prime order with 0, and, by analogy, the same holds for J_2. However, if J_1 (or J_2) is a zero semigroup, it cannot be a proper subsemigroup of S. Thus, either S is a two-element zero semigroup or both J_1 and J_2 are as stated in (i$_1$). This completes the proof of the theorem.

The following result can be considered as a semigroup theoretical analogue of a theorem due to R. Wiegandt [5].

Theorem 2. *Let S be a semigroup with* 0. *Then the following conditions are equivalent:*

(i$_2$) *S is either the* 0*-direct union of two subgroups with* 0 *of S or a two-element zero semigroup;*

(ii$_2$) (0) *is not a prime ideal of S, but every proper quasi-ideal of S is a subgroup with* 0 *of S;*

(iii$_2$) *S contains divisors of* 0 *but every proper quasi-ideal of S is free from divisors of* 0;

(iv$_2$) (0) *is not a prime ideal of S, but it is prime in every proper quasi-ideal of S.*

Remark 2. Let $S=\{0, a, e\}$ be a subsemigroup with 0 having the following Cayley table:

	0	a	e
0	0	0	0
a	0	a	a
e	0	a	e

Then S is not a group with 0 but every proper quasi-ideal of S ($\{0\}$ and $\{0, a\}$) is either a subgroup or a subgroup with 0 of S. This example shows that neither condition (ii$_1$) nor R. Wiegandt's condition (d) in the "Satz" of [5] has a word-for-word analogue in Theorem 2.

Proof. The implications (i$_2$)$\Rightarrow$(ii$_2$) and (ii$_2$)$\Rightarrow$(iii$_2$) are obvious.

(iii$_2$)$\Rightarrow$(iv$_2$). The same proof as for (iii$_1$)$\Rightarrow$(iv$_1$) goes through, because $a\cup Sa$ is not only a subsemigroup but also a quasi-ideal (in fact, a left ideal) of S.

(iv$_2$) $\Rightarrow$ (i$_2$). Let $J_1, J_2 \lhd S, J_1 J_2 = 0, J_1, J_2 \neq 0$. An argument analogous to that used in the proof of (iv$_1$)$\Rightarrow$(i$_1$) shows that J_1 (and J_2) cannot have non-trivial quasi-ideals whence it is either a two-element zero semigroup or a group, and the first case can hold only if $J_1=S$ ($J_2=S$, resp.). This completes the proof.

Our next theorem shows that conditions (c) and (g) in the mentioned "Satz" of Wiegandt are not equivalent to the other ones in the case of semigroups. However, the corollary exhibits the way of repairing the matter.

Theorem 3. *Let S be a semigroup with 0. Then the following conditions are equivalent:*

(i$_3$) *S is either the 0-direct union of two left 0-simple semigroups which are not zero semigroups or S itself is a two-element zero semigroup;*

(ii$_3$) *(0) is not a prime ideal of S but every proper left ideal of S is left 0-simple;*

(iii$_3$) *S contains divisors of 0 but every proper left ideal of S is free from divisors of 0;*

(iv$_3$) *(0) is not a prime ideal of S but it is prime in every proper left ideal of S.*

Proof. (i$_3$)$\Rightarrow$(ii$_3$) is obvious, and so is (ii$_3$)$\Rightarrow$(iii$_3$) because a left 0-simple semigroup is a left simple semigroup with 0 adjoined.

(iii$_3$)$\Rightarrow$(iv$_3$). The same as (iii$_1$)$\Rightarrow$(iv$_1$).

(iv$_3$) $\Rightarrow$ (i$_3$). Let $J_1, J_2 \lhd S, J_1 J_2 = 0, J_1, J_2 \neq 0$. Analogously to Theorems 1 and 2, here it follows that J_1 and J_2 cannot have non-trivial left ideals, and the same proof as there goes through with the obvious modifications.

Corollary 1. *Let S be a semigroup with 0. Then the following conditions are equivalent:*

(i_2) *S is either the 0-direct union of two subgroups with 0 of S or a two-element zero semigroup;*

(ii_c) (0) *is not a prime ideal of S but every proper one-sided ideal of S is a group;*

(iii_c) *S contains divisors of 0 but every proper one-sided ideal of S is free from divisors of* 0;

(iv_c) (0) *is not a prime ideal of S but it is prime in every proper one-sided ideal of S.*

This Corollary can be obtained by simply putting together Theorem 3 and its dual.

Remark 3. Corollary 1 yields a trivial proof of Theorem 2 according to the following scheme:

$$(i_2)\ \left\{\begin{array}{l}\Rightarrow (ii_2) \Rightarrow (ii_c) \Rightarrow \\ \Rightarrow (iii_2) \Rightarrow (iii_c) \Rightarrow \\ \Rightarrow (iv_2) \Rightarrow (iv_c) \Rightarrow\end{array}\right\}\ (i_2)$$

Nevertheless, we preferred to give there an independent proof, in order to preserve the natural order of the results.

Condition (i) in Wiegandt's "Satz" determines an even larger class of semigroups.

Theorem 4. *Let S be a semigroup with 0. Then the following conditions are equivalent:*

(i_4) *S is either the 0-direct union of two simple semigroups with 0 adjoined or 0-simple with divisors of* 0;

(iii_4) *S contains divisors of 0 but every proper ideal of S is free from divisors of* 0.

Proof. (i_4)$\Rightarrow$(iii_4) is obvious.

(iii_4)$\Rightarrow$(i_4). Let $a, b\in S$, $a\neq 0\neq b$, $ab=0$. Then $(a)\cup(b)=S$. Suppose S has a non-trivial ideal J. Then either $Ja=0$ or $bJ=0$ because $Ja\cdot bJ=0$ and J is free from divisors of 0. Let e.g. $Ja=0$. Then $a\notin J$ and $(a)\cup J=S$. Thus, either $b\in(a)$ or $b\in J$. In the first case $(a)=(a)\cup(b)=S$ whence $JS=Ja\cup JaS=0$ and $J=0$, contrary to the assumption. In the second case $Ja=0$ implies $ba=0$, and either $Jb=0$ or $aJ=0$. But $Jb\neq 0$ since $b\in J$. Thus, $aJ=Ja=0$, and also $(a)J=J(a)=0$ whence $((a)\cap J)^2=0$ and therefore $(a)\cap J=0$. We have obtained that S is the 0-direct union of (a) and J. Both must be 0-simple, else a proper ideal containing divisors of 0 could be found. However, (a) and J themselves do not contain divisors of 0, and so they are simple semigroups with 0 adjoined; q. e. d.

As in [5], the condition of type (iv) leads to a different class of semigroups. Here the analogy with the ring case is again complete.

Theorem 5. *Let* S *be a semigroup with* 0. *The following conditions are equivalent:*

(i_5) S *is either the* 0-*direct union of two simple semigroups which are not zero semigroups or* S *itself is the two-element zero semigroup;*

(iv_5) (0) *is not a prime ideal of* S *but it is prime in every proper ideal of* S.

The proof is essentially the same as that of (iv_3)$\Rightarrow$(i_3).

It is easy to find the corresponding conditions (ii_4) and (ii_5) which can be inserted in Theorems 4 and 5, respectively.

References

[1] L. Rédei, Die einstufig nichtreguläre Ringe, *Acta Sci. Math.*, **20** (1959), 238—244.

[2] O. Steinfeld, Die einstufig nichtregulären bzw. nichtprimen Ringe, *Acta Sci. Math.*, **22** (1961), 82—84.

[3] F. Szász, Ein kurzer elementarer Beweis eines Satzes von O. Steinfeld über die einstufig nichtprimen Ringe, *Periodica Math. Hung.*, **7** (1976), 9—10.

[4] R. Wiegandt, Bemerkung über die einstufig nichtregulären Ringe, *Acta Sci. Math.*, **21** (1960), 350—352.

[5] ——, Bemerkung zu einer Arbeit von Herrn Steinfeld, *Acta Sci. Math.*, **23** (1962), 74—75.

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
P. O. BOX 127
1364 BUDAPEST, HUNGARY

Acta Sci. Math., 47 (1984), 61—70

Basic permutation groups on infinite sets

LÁSZLÓ SZABÓ

1. Introduction

An algebra $\mathfrak{A}=\langle A; F\rangle$ is said to be (locally) complete if every finitary operation on A is a (local) term function of $\mathfrak{A}$, and $\mathfrak{A}$ is (locally) functionally complete if every finitary operation on A is a (local) algebraic function of $\mathfrak{A}$. Following SALOMAA [11], a permutation group G on a finite set A is said to be basic if for any surjective finitary operation f on A depending on at least two variables, the algebra $\langle A; G\cup\{f\}\rangle$ is complete. For example the full symmetric group on A if $|A|\geqq 5$ (SALOMAA [10]), the triply transitive permutation groups G on A if $|A|\geqq 4$ and G is affine with respect to no elementary abelian 2-group (SCHOFIELD [12]), and the doubly transitive permutation groups G on A if $|A|\geqq 3$ and G is affine with respect to no abelian elementary p-group, p prime (KNOEBEL [5]), are basic groups.

There is an interesting analogy between the above mentioned examples and some results on the functional completeness of finite algebras with large automorphism groups. The first example is the counterpart of the result of CSÁKÁNY [1] stating that almost every nontrivial finite algebra whose automorphism group is the full symmetric group is functionally complete; up to equivalence there are only six exceptions. The second and third examples correspond to the results of [14] and [6], respectively, stating that except for some affine spaces every nontrivial finite algebra whose automorphism group is triply, resp., doubly transitive, is functionally complete. A result in [7] completes this series stating that every nontrivial finite algebra whose automorphism group is a basic group, is functionally complete.

For infinite sets the full symmetric groups are not basic groups in Salomaa's sense (replacing the completeness by local completeness) as it was shown in [9] and [5], but if f is a nontrivial idempotent operation on an infinite set A and S_A is the full symmetric group on A, then the algebra $\langle A; S_A\cup\{f\}\rangle$ is locally complete ([9]). Therefore in this paper a permutation group G on an infinite set A is said

Received November 15, 1982.

to be basic if for any nontrivial idempotent operation f on A, the algebra $\langle A; G\cup\{f\}\rangle$ is locally complete. (Remark that for finite sets this definition yields exactly the basic groups in Salomaa's sense.) The best result for basic groups on infinite sets was given in [9]: every triply transitive permutation group on an infinite set is basic if it is affine with respect to no elementary abelian 2-group.

As in the case of finite sets, we can observe the same analogy between the above mentioned basic permutation groups on infinite sets and some results on locally functionally complete algebras whose automorphism group is basic. Namely, if the automorphism group of an infinite algebra $\mathfrak{A}$ is the full symmetric group or a triply transitive permutation group which is affine with respect to no elementary abelian 2-group, then $\mathfrak{A}$ is locally functionally complete (FRIED, KAISER and MÁRKI [2]; KAISER and MÁRKI [4]).

In [4] the following is proved: If $\mathfrak{A}=\langle A; F\rangle$ is an infinite algebra whose automorphism group G is doubly primitive, the stabilizer G_a of any element $a\in A$ admits no partial order on the set $A\setminus\{a\}$, and G is affine with respect to no elementary abelian 2-group, then $\mathfrak{A}$ is locally functionally complete. Observing these results the following question is naturally arising: Is every permutation group G on an infinite set having the properties mentioned above a basic group? The aim of this paper is to give an affirmative answere to this question (Theorem 8 and Corollary 9).

This paper was partially prepared during the author's three month stay at CRMA, Université de Montréal. The author is indebted to Professor I. G. Rosenberg for the invitation. The support provided by ENRSC Canada grant A-9128 and Ministère de l'Education du Québec FCAC grant E-539 is gratefully acknowledged.

2. Preliminaries

Let A be a nonempty set. The set of n-ary operations (or functions) on A will be denoted by $O_A^{(n)}$ $(n\geqq 1)$ and we set $O_A=\bigcup_{n=1}^{\infty} O_A^{(n)}$. An operation $f\in O_A$ is idempotent if we have $f(a, ..., a)=a$ for every $a\in A$. f is trivial if it is a projection. A ternary operation $d\in O_A^{(3)}$ is called a *majority function* if for all $x, y\in A$ we have $d(x, x, y)=d(x, y, x)=d(y, x, x)=x$. An operation $t\in O_A^{(3)}$ is said to be a *minority function* if $t(x, y, y)=t(y, x, y)=t(y, y, x)=x$ for all $x, y\in A$. By an n-ary i-th semi-projection $(3\leqq n,\ 1\leqq i\leqq n)$ we mean an operation $q\in O_A^{(n)}$ which has the following property: $q(x_1, ..., x_n)=x_i$ whenever at least two elements among $x_1, ..., x_n$ are equal.

We adopt the terminology of [3] except that polynomials will be called *term functions*. For an algebra $\mathfrak{A}=\langle A; F\rangle$, the sets of term functions and of algebraic functions of $\mathfrak{A}$ will be denoted by $T(\mathfrak{A})$ and $A(\mathfrak{A})$, respectively. An operation

$f\in O_A^{(n)}$ is said to be a *local term (algebraic) function* of $\mathfrak{A}$ if for every finite $B\subseteq A^n$ there is a $g\in T(\mathfrak{A})\cap O_A^{(n)}$ $(g\in A(\mathfrak{A})\cap O_A^{(n)})$ such that $f|B=g|B$. The sets of local term functions and of local algebraic functions of $\mathfrak{A}$ will be denoted by $\hat{T}(\mathfrak{A})$ and $\hat{A}(\mathfrak{A})$, respectively. An algebra $\mathfrak{A}=\langle A; F\rangle$ is termed *locally complete (locally functionally complete)* if $\hat{T}(\mathfrak{A})=O_A$ $(\hat{A}(\mathfrak{A})=O_A)$. $\mathfrak{A}$ is called *trivial* if $T(\mathfrak{A})$ contains projections only.

If ϱ is an h-ary relation on A, i.e., $\varrho\subseteq A^h$, then $\operatorname{Pol}\varrho$ denotes the set of all operations from O_A preserving ϱ. A binary relation ϱ on A is *locally bounded* if for every finite $B\subseteq A$ we have $B\times\{u\}\subseteq\varrho$ and $\{v\}\times B\subseteq\varrho$ for some $u, v\in A$. For $2\leq h$ let

$$\sigma_h=\{(a_1, \ldots, a_h)\in A^h: a_i\neq a_j,\quad 1\leq i<j\leq h\}.$$

Furthermore, we set $\iota_h=A^h\setminus\sigma_h$. An h-ary relation ϱ on A $(h\geq 2)$ is *totally reflexive* if $\iota_h\subseteq\varrho$. ϱ is *totally symmetric* if $(a_1, \ldots, a_h)\in\varrho$ implies $(a_{1\pi}, \ldots, a_{h\pi})\in\varrho$ for every permutation π of $\{1, \ldots, h\}$. If ϱ is totally reflexive, totally symmetric, $\varrho\neq A^h$, and to every finite $B\subseteq A$ there is a $u\in A$ such that $B^{h-1}\times\{u\}\subseteq\varrho$ then ϱ is called *locally central*. The proper unary relations are also referred to as locally central relations. A binary relation ϱ is *reflexive, areflexive, symmetric,* and *asymmetric* if $\iota_2\subseteq\varrho$, $\iota_2\cap\varrho=\emptyset$, $\varrho=\varrho^{-1}$ and $\varrho\cap\varrho^{-1}=\emptyset$. If $f\in O_A^{(n)}$ then $f^\bullet=\{(a_1, \ldots, a_n, f(a_1, \ldots, a_n))\in A^{n+1} : a_1, \ldots, a_n\in A\}$.

Now we are ready to formulate the local completeness criterion from [9].

Theorem 1. *Let $\mathfrak{A}=\langle A; F\rangle$ be an algebra. Then $\mathfrak{A}$ is locally complete if $T(\mathfrak{A})\subseteq\operatorname{Pol}\varrho$ for no relation ϱ of one of the following types:*

(1) *locally bounded partial orders,*

(2) *nontrivial equivalence relations,*

(3) *binary relations $s^\bullet$ where s is a fixed point free permutation of A whose cycles are either all of the same prime length or all infinite,*

(4) *reflexive, symmetric relations ϱ with $\bigcup_{n=1}^{\infty}\varrho^n=A^2$ and $\varrho^n\neq A^2$, $n=1, 2, \ldots$ $(\varrho^1=\varrho,\ \varrho^{n+1}=\varrho^n\circ\varrho)$,*

(5) *binary locally bounded reflexive antisymmetric relations ϱ with $\varrho^2=A^2$,*

(6) *binary locally bounded areflexive symmetric relations,*

(7) *binary locally bounded areflexive asymmetric relations,*

(8) *ternary relations $\varrho=\sigma\cup\Delta_{12}$ with $\emptyset\neq\sigma\subseteq\sigma_3$ and $\Delta_{12}=\{(a,a,b)\in A^3: a,b\in A\}$ such that for all $x, y, z, t\in A$, $(x, y, z)\in\varrho$ implies $(y, x, z)\in\varrho$, $(x, t, z)\in\varrho$ and $(y, t, z)\in\varrho$ imply $(x, y, z)\in\varrho$, and for every finite $B\subseteq A$ we have $B^2\times\{u\}\subseteq\varrho$ for some $u\in A$,*

(9) *quaternary relations $m^\bullet$ where $m(x, y, z)=x-y+z$ for all $x, y, z\in A$ and $\langle A; +\rangle$ is an abelian group which is either an elementary p-group or a torsion-free divisible group,*

(10) *locally central relations,*

(11) *totally reflexive and totally symmetric h-ary relations* ϱ *with* $\varrho \neq A^h$ *and* $h \geqq 3$, *that are not locally central.*

We also need the following result from [9]:

Proposition 2. $\mathrm{Pol}\,\sigma_2$ *and* $\mathrm{Pol}\,(\sigma_3 \cup \Delta_{12})$ *contain no nontrivial idempotent operations.* (Here $\sigma_3 \cup \Delta_{12}$ is the greatest relation of type (8).)

An operation $f \in O_A^{(n)}$ is said to be *affine* with respect to an abelian group $\langle A; +\rangle$ if for all $x_1, y_1, ..., x_n, y_n \in A$ we have

$$f(x_1+y_1, ..., x_n+y_n) = f(x_1, ..., x_n)+f(y_1, ..., y_n)-f(0, ..., 0).$$

It is easy to check that this means exactly that $f \in \mathrm{Pol}\, m^{\bullet}$ where $m(x, y, z)=x-y+z$ for all $x, y, z \in A$. A set of operations $F \subseteqq O_A$ as well as the algebra $\langle A; F\rangle$ is said to be affine with respect to $\langle A; +\rangle$ if every operation in F is affine with respect to $\langle A; +\rangle$.

The full symmetric group acting on A will be denoted by S_A. For a permutation group $G \leqq S_A$, a subset B of A is called a *block* of G if for every $\pi \in G$, either $B\pi = B$ or $B\pi \cap B = \emptyset$. The one element subsets $\{a\}$, $a \in A$, and A are *trivial blocks* of G. A transitive permutation group $G \leqq S_A$ is said to be *primitive* if it has trivial blocks only. (Notice that except for the case $|A|=2$, the latter condition implies the transitivity of G.) We shall often use the following equivalent definition: for $|A| \geqq 3$, a permutation group $G \leqq S_A$ is primitive if $G \subseteqq \mathrm{Pol}\,\varrho$ for no nontrivial equivalence relation ϱ on A, and for $|A|=2$ if $G=S_A$. G is *doubly primitive* if it is doubly transitive and the stabilizer G_a of any element $a \in A$ is primitive on the set $A \setminus \{a\}$.

A permutation group G on an infinite set A is said to be a *basic group* if for every nontrivial idempotent operation f on A, the algebra $\langle A; G \cup \{f\}\rangle$ is locally complete.

We shall use the following well-known fact for primitive permutation groups:

Proposition 3 ([13; Theorem 10.5.7]). *If* $G \leqq S_A$ *is transitive and* $a \in A$, *then* G *is primitive if and only if* G_a *is a maximal proper subgroup of* G.

Finally we need the following well-known result.

Proposition 4 (see e.g. [6]). *If an algebra* $\mathfrak{A}$ *has a nontrivial idempotent term function then it has a majority function or a minority function or a nontrivial first semi-projection or a nontrivial binary idempotent operation among its term functions.*

3. Results

From now on A is supposed to be an infinite set.

Lemma 5. *If G is a doubly primitive permutation group on A and $G \subseteq \mathrm{Pol}\,\varrho$ where ϱ is one of the relations of types* (1)—(11), *then $\varrho=\sigma_2$ or $\varrho=\sigma_3 \cup \Delta_{12}$ or ϱ is an at least ternary totally reflexive totally symmetric relation or a relation $m^{\bullet}$ where $m(x, y, z)=x-y+z$ for all $x, y, z \in A$ and $\langle A; +\rangle$ is an elementary 2-group.*

Proof. Let ϱ be one of the relations of types (1)—(11) and suppose that $G \subseteq \mathrm{Pol}\,\varrho$. Clearly ϱ cannot be a unary relation (of type (10)). If ϱ is binary then $\varrho=\sigma_2$ since G is doubly transitive. If ϱ is an at least ternary relation of type (10) or a relation of type (11) then it is totally reflexive and totally symmetric. Now let ϱ be a relation of type (8) and let $a \in A$ be an arbitrary element. Define a binary relation θ_a on $A \setminus \{a\}$ as follows: $(x, y) \in \theta_a$ if and only if $(x, y, a) \in \varrho$. Taking into consideration the properties of ϱ we immediately get that θ_a is an equivalence relation. Furthermore, since $G \subseteq \mathrm{Pol}\,\varrho$, every permutation in G_a preserves θ_a. Therefore, owing to the primitivity of G_a on $A \setminus \{a\}$, the relation θ_a is either the equality or the full relation on $A \setminus \{a\}$. By the definition of ϱ, $\varrho \cap \sigma_3 \neq \emptyset$. Therefore there are pairwise distinct elements $a_1, a_2, a_3 \in A$ such that $(a_1, a_2, a_3) \in \varrho$. Choose a permutation $\pi \in G$ such that $a_3\pi=a$. Then $(a_1, a_2, a_3) \in \varrho$ implies $(a_1\pi, a_2\pi, a)=(a_1\pi, a_2\pi, a_3\pi) \in \varrho$ and $(a_1\pi, a_2\pi) \in \theta_a$, showing that θ_a is not the equality relation. Hence θ_a is the full relation, i.e., for all $x, y \in A \setminus \{a\}$ we have $(x, y, a) \in A$. Since a was chosen arbitrarily, it follows that $\varrho=\sigma_3 \cup \Delta_{12}$.

Finally, let $\varrho=m^{\bullet}$ where $m(x, y, z)=x-y+z$ for all $x, y, z \in A$ and $\langle A; +\rangle$ is an abelian group which is either an elementary p-group or a torsionfree divisible group. Let 0 be the unit element of $\langle A; +\rangle$. If $\pi \in G \subseteq \mathrm{Pol}\, m^{\bullet}$ then for all $x \in A$ we have $x\pi=(x\pi-0\pi)+0\pi$ and the map $x \to x\pi-0\pi$ from A to A is an automorphism of $\langle A; +\rangle$. Indeed, if $a, b \in A$ then $(a, 0, b, a+b) \in m^{\bullet}$ implies $(a\pi, 0\pi, b\pi, (a+b)\pi) \in m^{\bullet}$. Thus $a\pi-0\pi+b\pi=(a+b)\pi$ and $(a+b)\pi-0\pi=(a\pi-0\pi)+(b\pi-0\pi)$. Hence we get that every permutation in G is of form $xr+c$ where $r\colon A \to A$ is an automorphism of $\langle A; +\rangle$ and $c \in A$. Therefore, the stabilizer G_0 consists of permutations of the form xr, $r \in \mathrm{Aut}\,\langle A; +\rangle$.

First let $\langle A; +\rangle$ be a torsionfree divisible group and define a binary relation θ on $A \setminus \{0\}$ as follows: $(x, y) \in \theta$ if and only if $mx=ny$ for some natural numbers m, n. Then θ is an equivalence relation preserved by every permutation in G_0. Therefore, owing to the primitivity of G_0, θ is the full relation on $A \setminus \{a\}$. Therefore $\langle A; +\rangle$ is isomorphic to the additive group of rational numbers. Furthermore it is easy to check that the automorphism group of the additive group of rational numbers is not primitive on the set of nonzero rational numbers. Thus G_0 is not primitive, contrary to our assumption.

Now let $\langle A; +\rangle$ be an elementary p-group (p prime), and define a binary relation Ψ on $A\setminus\{0\}$ as follows: $(x, y)\in\Psi$ if and only if $x=my$ for some natural number m. Then Ψ is an equivalence relation and every permutation in G_0 preserves Ψ. Therefore, since G_0 is primitive, Ψ is either the equality or the full relation on $A\setminus\{a\}$. In the first case $\langle A; +\rangle$ is an elementary 2-group. In the second case we have $A=\{0, a, 2a, ..., (p-1)a\}$ where $a\in A$ with $a\neq 0$. This contradicts our assumption on A (A is infinite).

Lemma 6. *If G is a doubly transitive permutation group on A and $d\in O_A^{(3)}$ is a majority function then the algebra $\langle A; G\cup\{d\}\rangle$ is locally complete.*

Proof. Taking into consideration Theorem 2, we have to prove that $G\cup\{d\}\subseteq\subseteq\mathrm{Pol}\,\varrho$ for no relation ϱ of types (1)—(11). Let ϱ be one of the relations listed in Theorem 1, and suppose $G\cup\{d\}\subseteq\mathrm{Pol}\,\varrho$. Making use of the fact that G is doubly transitive, we immediately get that ϱ cannot be a relation of types (1), (2), (3), (4), (5), (7) or a unary or binary relation of type (10). Furthermore, if ϱ is of type (6) then $\varrho=\sigma_2$ and, by Proposition 2, $d\notin\mathrm{Pol}\,\sigma_2$. If ϱ is of type (8), then, by definition, $(a, b, c)\in\varrho$ for some pairwise distinct elements $a, b, c\in A$ and $(a, a, b)\in\varrho$, $(b, b, b)\in\varrho$. It follows $(a, b, b)=(d(a, a, b),\ d(b, a, b),\ d(c, b, b))\in\varrho$, contrary to the definition of ϱ. If ϱ is an h-ary ($h\geqq 3$) relation of type (10) or (11) and $(a_1, ..., a_h)\in A^h\setminus\varrho$ then $(a_2, a_2, a_3, a_4, ..., a_h)\in\varrho$, $(a_1, a_1, a_3, a_4, ..., a_h)\in\varrho$ and $(a_1, a_2, a_2, a_4, ..., a_h)\in\varrho$ imply that $(a_1, a_2, a_3, a_4, ..., a_h)=(d(a_2, a_1, a_1), d(a_2, a_1, a_2), d(a_3, a_3, a_2), d(a_4, a_4, a_4), ..., d(a_h, a_h, a_h))\in\varrho$, a contradiction.

Finally, let $\varrho=m^\bullet$ where $m(x, y, z)=x-y+z$ for all $x, y, z\in A$ and $\langle A; +\rangle$ is an abelian group. Let $0\in A$ be the unit element of $\langle A; +\rangle$ and choose a non-unit element $a\in A$. Then $(0, 0, a, a)\in m^\bullet$, $(0, -a, 0, a)\in m^\bullet$ and $(a, 0, 0, a)\in m^\bullet$ imply $(0, 0, 0, a)=(d(0, 0, a),\ d(0, -a, 0),\ d(a, 0, 0),\ d(a, a, a))\in m^\bullet$ and $0==0-0+0=a$, contrary to the choice of a.

Lemma 7. *If G is a primitive permutation group on A and $t\in O_A^{(3)}$ is a minority function then the algebra $\langle A; G\cup\{t\}\rangle$ is either locally complete or affine with respect to an elementary 2-group.*

Proof. If $\langle A; G\cup\{t\}\rangle$ is locally incomplete then, by Theorem 2, $G\cup\{t\}\subseteq\mathrm{Pol}\,\varrho$ where ϱ is one of the relations of types (1)—(11). Our proof will be complete if we show that ϱ is of type (9) determined by an elementary 2-group.

If ϱ is a binary reflexive relation then ϱ is an equivalence relation. Indeed, $(a, b)\in\varrho$ implies $(b, a)=(t(b, a, a), t(b, a, b))\in\varrho$ and $(a, b)\in\varrho$, $(b, c)\in\varrho$ imply $(a, c)=(t(a, b, b),\ t(b, b, c))\in\varrho$. Since G is primitive, $G\subseteq\mathrm{Pol}\,\varrho$ implies that $\varrho=\iota_2$ or $\varrho=A^2$, a contradiction. If ϱ is a locally bounded areflexive relation and $(a, b)\in\varrho$ then there is a $c\in A$ such that $(a, c)\in\varrho$ and $(b, c)\in\varrho$. Therefore $(b, b)=(t(a, a, b),\ t(b, c, c))\in\varrho$, contrary to the areflexivity of ϱ. If $\varrho=s^\bullet$ where

s is a permutation on A then $G\in \mathrm{Pol}\, s^\bullet$ implies that every cycle of any power s^n of s $(n=1, 2, \ldots)$ is a block of G, showing that s is the identity permutation. Hence ϱ cannot be a relation of types (1)—(7) or a unary or binary relation of type (10). (The unary relations are excluded by the transitivity of G.)

If ϱ is a relation of type (8) then, by definition, there are pairwise different elements $a, b, c\in A$ such that $(a, b, c)\in\varrho$. Now $(a, b, c)\in\varrho$, $(a, a, c)\in\varrho$ and $(b, b, b)\in\varrho$ imply $(b, a, b)=(t(a, a, b),\ t(b, a, b),\ t(c, c, b))\in\varrho$, contrary to the definition of ϱ. If ϱ is an h-ary $(h\geqq 3)$ relation of type (10) or (11) and $(a_1, \ldots, a_h)\in A^h\setminus\varrho$ then $(a_1, a_1, a_3, \ldots, a_h)\in\varrho$, $(a_3, a_2, a_3, \ldots, a_h)\in\varrho$ and $(a_3, a_1, a_3, \ldots, a_h)\in\varrho$ imply $(a_1, a_2, a_3, \ldots, a_h)=(t(a_1, a_3, a_3), t(a_1, a_2, a_1), t(a_3, a_3, a_3), \ldots, t(a_h, a_h, a_h))\in\varrho$, contrary to the choice of $(a_1, \ldots, a_h)$.

Hence $\varrho=m^\bullet$ where $m(x, y, z)=x-y+z$ for all $x, y, z\in A$ and $\langle A; +\rangle$ is an abelian group. Now for every $a\in A$ we have $(a, 0, a, 2a)\in m^\bullet$, $(a, a, 0, 0)\in m^\bullet$ and $(0, a, a, 0)\in m^\bullet$, which imply that $(0, 0, 0, 2a)=(t(a, a, 0),\ t(0, a, a),\ t(a, 0, a),\ t(2a, 0, 0))\in m^\bullet$, i.e., $2a=0-0+0=0$. Therefore $\langle A; +\rangle$ is an elementary 2-group, which completes the proof.

Theorem 8. *Let G be a doubly primitive permutation group on A such that for every $a\in A$, the stabilizer G_a admits no nontrivial partial order on the set $A\setminus\{a\}$. Then for every nontrivial idempotent function $f\in O_A$, the algebra $\mathfrak{A}=\langle A; G\cup\{f\}\rangle$ is either locally complete or affine with respect to an elementary 2-group.*

Proof. If $\mathfrak{A}$ has a majority function or a minority function among its term functions then our claim follows from Lemma 6 or Lemma 7. Therefore, suppose that $\mathfrak{A}$ has neither majority functions nor minority functions among its term functions. Then, by Proposition 4, $\mathfrak{A}$ has a nontrivial first semi-projection or a nontrivial binary idempotent operation among its term functions. If $\mathfrak{A}$ is locally incomplete then, by Theorem 1, $G\cup\{f\}\subseteq \mathrm{Pol}\,\varrho$ where ϱ is one of the relations of types (1)—(11). Taking into consideration Lemma 5 and Proposition 2, we immediately get that ϱ is an at least ternary totally reflexive and totally symmetric relation or $\varrho=m^\bullet$ where $m(x, y, z)=x-y+z$ for all $x, y, z\in A$ and $\langle A; +\rangle$ is an elementary 2-group. Therefore our proof will be complete if we show that $G\cup\{f\}\subseteq \mathrm{Pol}\,\varrho$ for no h-ary $(h\geqq 3)$ totally reflexive and totally symmetric relation different from A^h. In order to prove this, consider the algebra $\mathfrak{B}=\langle A; I\rangle$ where I is the set of all idempotent term functions of $\mathfrak{A}$. We show that $\mathfrak{B}$ has the $1^1/_2$-interpolation property, i.e., for all $a, b, c\in A$, $a\neq b$, there is a unary algebraic function g of $\mathfrak{B}$ such that $g(a)=a$ and $g(b)=c$. Then, clearly, $\mathfrak{B}$ also has the 2-interpolation property, i.e., for all $a, b, c, d\in A$, $a\neq b$, there is a unary algebraic function g of $\mathfrak{B}$ such that $g(a)=c$ and $g(b)=d$.

For $a, b\in A$, $a\neq b$, let $B(a, b)$ denote the set of elements $c\in A$ for which there is a unary algebraic function g of $\mathfrak{A}$ such that $g(a)=a$ and $g(b)=c$. We have to show that $B(a,b)=A$ for all $a,b\in A$, $a\neq b$. It is easy to check that $a,b\in B(a,b)$ and $c\in B(a,b)$ imply $B(a,c)\subseteq B(a,b)$. Furthermore, for every $\pi\in G$, $B(a,b)\pi=$ $=B(a\pi,b\pi)$. To show the second statement let $c\in B(a,b)$ and let g be a unary algebraic function of $\mathfrak{B}$ such that $g(a)=a$ and $g(b)=c$. Then, clearly, there is an n-ary term function t of $\mathfrak{B}$ and there are elements $a_2,\ldots,a_n\in A$ such that $g(x)=t(x,a_2,\ldots,a_n)$ for all $x\in A$. Then, by definition, the n-ary operation $t'(x_1,\ldots,x_n)=(t(x_1\pi^{-1},\ldots,x_n\pi^{-1}))\pi$ is a term function of $\mathfrak{B}$ and for the unary algebraic function $g'(x)=t'(x,a_2\pi,\ldots,a_n\pi)$ we have $g'(a\pi)=a\pi$ and $g'(b\pi)=c\pi$. It follows that $B(a,b)\pi\subseteq B(a\pi,b\pi)$. Repeating this reasoning for $a\pi, b\pi$, and π^{-1} instead of a,b and π we have $B(a\pi,b\pi)\pi^{-1}\subseteq B(a,b)$ and $B(a\pi,b\pi)\subseteq$ $\subseteq B(a,b)\pi$. Hence $B(a,b)\pi=B(a\pi,b\pi)$. Taking into consideration the double transitivity of G, we immediately get that $B(a,b)$ has the same cardinality for all $a,b\in A$, $a\neq b$.

Now we show that $|B(a,b)|\geqq 3$ for all $a,b\in A$, $a\neq b$. First suppose that $\mathfrak{B}$ has an n-ary $(n\geqq 3)$ nontrivial first semi-projection q among its term functions. Since q is not the first projection, there are pairwise different elements $x_1,\ldots,x_n\in A$ such that $q(x_1,\ldots,x_n)=x_0\neq x_1$. We may suppose that $x_0\neq x_2$. Then for the unary algebraic function $g(x)=q(x_1,x,x_3,\ldots,x_n)$ we have $g(x_1)=x_1$ and $g(x_2)=x_0$. Thus $x_0,x_1,x_2\in B(x_1,x_2)$, showing that $|B(x_1,x_2)|\geqq 3$. Now suppose that $\mathfrak{B}$ has a nontrivial binary idempotent term function and denote it by juxtaposition. If $ab=c\notin\{a,b\}$ for some $a,b\in A$, $a\neq b$ then for the unary algebraic function $g(x)=ax$ we have $g(a)=a$ and $g(b)=c$. Thus $a,b,c\in B(a,b)$ and $|B(a,b)|\geqq 3$. Suppose that $xy\in\{x,y\}$ for all $x,y\in A$. First consider the case when there is a zero element, i.e., there is an element $a\in A$ such that $a=ax=xa$ for all $x\in A$. Then choose two different elements $b,c\in A\setminus\{a\}$. We may assume that $bc=c$. Thus the unary algebraic function $g(x)=xc$ shows that $a,b,c\in B(a,b)$ and $|B(a,b)|\geqq 3$. Finally consider the case when there is no zero element and let $a,b\in A$, $a\neq b$, $ab=a$. If $ac=c$ for some $c\in A$, $c\neq a$, then $g(x)=ax$ shows that $c,b,a\in B(c,b)$ and $|B(c,b)|\geqq 3$. If $ax=a$ for all $x\in A$, then since our operation is not the first projection, $bc=c$ for some $b,c\in A$, $b\neq c$. If $xc=c$ for all $x\in A$ then we have $a=ac=c$ and a is a zero element, contrary to our assumption. Therefore $dc=d$ for some $d\in A$, $d\neq c$. Thus $g(x)=xc$ shows $d,b,c\in B(d,b)$ and $|B(d,b)|\geqq 3$.

Let a and b be two arbitrary different elements and define a binary relation σ on the set $A\setminus\{a\}$ as follows: $x\,\sigma\,y$ if and only if $B(a,x)\subseteq B(a,y)$. Then σ is a reflexive transitive relation. Furthermore, if $\pi\in G_a$ and $x\,\sigma\,y$ then $B(a,x)\subseteq$ $\subseteq B(a,y)$ implies $B(a,x\pi)=B(a,x)\pi\subseteq B(a,y)\pi=B(a,y\pi)$ and $x\pi\,\sigma\,y\pi$. Hence every permutation in G_a preserves σ. Clearly, σ is not the equality relation on $A\setminus\{a\}$. Indeed, if $c\in B(a,b)$, $c\neq a,b$, then $B(a,c)\subseteq B(a,b)$ shows that $c\,\sigma\,b$. By the assumption on G_a, σ cannot be a partial order, therefore we have $b_1\,\sigma\,b_2$ and $b_2\,\sigma\,b_1$ for some $b_1,b_2\in A\setminus\{a\}$, $b_1\neq b_2$. Thus, by definition, $B(a,b_1)=$

$=B(a, b_2)$. Choose a permutation $\varphi \in G_a$ such that $b_1\varphi = b$. Then we have $B(a, b) = B(a\varphi, b_1\varphi) = B(a, b_1)\varphi = B(a, b_2)\varphi = B(a, b_2\varphi)$. Put $b' = b_2\pi$. Clearly $b \neq b'$. If $\pi \in G_{a,b}$ then we have $B(a, b)\pi = B(a\pi, b\pi) = B(a, b)$. Let $\tau \in G_a$ be a permutation such that $b\tau = b'$. Then $B(a, b)\tau = B(a\tau, b\tau) = B(a, b') = B(a, b)$ and $B(a, b)\tau^{-1} = B(a, b')\tau^{-1} = B(a\tau^{-1}, b'\tau^{-1}) = B(a, b)$. Let us denote by G' the subgroup of G_a generated by the set $G_{a,b} \cup \{\tau, \tau^{-1}\}$. The above argument shows that $B(a, b)\pi = B(a, b)$ for all $\pi \in G'$. Since G_a is a primitive permutation group on $A \setminus \{a\}$, by Proposition 3, $G_{a,b}$ is a maximal subgroup in G_a. Therefore, $G_{a,b} \subset G' \subseteqq G_a$ implies $G' = G_a$. Thus we get that the set $B(a, b) \setminus \{a\}$ is a block of G_a. Again by the primitivity of G_a we have $B(a, b) \setminus \{a\} = \{b\}$ or $B(a, b) \setminus \{a\} = A \setminus \{a\}$. The first case cannot occur since $|B(a, b)| \geqq 3$. Hence $B(a, b) = A$. Since a and b were chosen arbitrarily, this means by definition exactly that the algebra $\mathfrak{B}$ has the $1^1/_2$-interpolation property and consequently the 2-interpolation property.

Now we are ready to prove that $G \cup \{f\} \subseteqq \operatorname{Pol} \varrho$ for no h-ary ($h \geqq 3$) totally reflexive and totally symmetric relation ϱ different from A^h. Let ϱ be an h-ary ($h \geqq 3$) totally reflexive and totally symmetric relation different from A^h such that $G \cup \{f\} \subseteqq \operatorname{Pol} \varrho$. Define a ternary relation $\hat{\varrho}$ as follows: $(a_1, a_2, a_3) \in \hat{\varrho}$ if and only if $(a_1, a_2, a_3, x_4, \ldots, x_h) \in \varrho$ for all $x_4, \ldots, x_h \in A$. Then, clearly, $\hat{\varrho}$ is a totally reflexive and totally symmetric relation with $\hat{\varrho} \neq A^3$. We show that if $g \in \operatorname{Pol} \varrho$ is a surjective operation then $g \in \operatorname{Pol} \hat{\varrho}$. Let n be the arity of g and let $(a_{i1}, a_{i2}, a_{i3}) \in \hat{\varrho}, i = 1, \ldots, n$. We have to show that $(g(a_{11}, \ldots, a_{n1}), g(a_{12}, \ldots, a_{n2}), g(a_{13}, \ldots, a_{n3})) \in \hat{\varrho}$, i.e., $(g(a_{11}, \ldots, a_{n1}), g(a_{12}, \ldots, a_{n2}), g(a_{13}, \ldots, a_{n3}), x_n, \ldots, x_h) \in \varrho$ for all $x_4, \ldots, x_h \in A$. Let $a_4, \ldots, a_h$ be arbitrary elements. Since g is surjective, $a_j = g(a_{1j}, \ldots, a_{nj})$ for some $a_{1j}, \ldots, a_{nj} \in A$, $j = 4, \ldots, h$. By definition $(a_{i1}, a_{i2}, a_{i3}, a_{i4}, \ldots, a_{ih}) \in \varrho$, $i = 1, \ldots, n$. Therefore we have $(g(a_{11}, \ldots, a_{n1}), g(a_{12}, \ldots, a_{n2}), g(a_{13}, \ldots, a_{n3}), a_4, \ldots, a_h) = (g(a_{11}, \ldots, a_{n1}), \ldots, g(a_{1h}, \ldots, a_{nh})) \in \varrho$ as required. Making use of this fact we immediately get that $G \cup \{f\} \subseteqq \operatorname{Pol} \hat{\varrho}$ and $I \subseteqq \operatorname{Pol} \hat{\varrho}$. Then $\hat{\varrho} \neq \iota_3$ since $\operatorname{Pol} \iota_3$ is known to consist of all unary operations and all operations taking on at most two values (see e.g. [8]) and I contains nontrivial idempotent operations.

Choose an element a from A arbitrarily and define a binary relation $\hat{\varrho}_a$ on A as follows: $(x, y) \in \hat{\varrho}_a$ if and only if $(x, y, a) \in \hat{\varrho}$. Taking into consideration the idempotency of the operations in I, it is easy to check that $I \subseteqq \operatorname{Pol} \hat{\varrho}_a$. Furthermore, $\hat{\varrho}_a \neq \iota_2$. Indeed, since $\hat{\varrho} \neq \iota_3$, there are pairwise different elements $c, d, e \in A$ such that $(c, d, e) \in \hat{\varrho}$. Choose a permutation $\pi \in G$ such that $e\pi = a$. Then we have $(c\pi, d\pi, a) = (c\pi, d\pi, e\pi) \in \hat{\varrho}$ and $(c\pi, d\pi) \in \hat{\varrho}_a$. Since $\hat{\varrho}_a$ is reflexive, every algebraic function of the algebra $\mathfrak{B} = \langle A; I \rangle$ preserves $\hat{\varrho}_a$. Making use of the fact that $\mathfrak{B}$ has the 2-interpolation property, we immediately get that $\hat{\varrho}_a = A^2$ and $A^2 \times \{a\} \subseteqq \hat{\varrho}$. Since a was chosen arbitrarily, it follows $\hat{\varrho} = A^3$, which is a contradiction, completing the proof of the theorem.

Corollary 9. *Let G be a doubly primitive permutation group on A such that for every $a \in A$ the stabilizer G_a admits no nontrivial partial order on the set $A \setminus \{a\}$. If G is affine with respect to no elementary 2-group then G is a basic group.*

Finally we give an example for a permutation group satisfying the assumption given in Theorem 8, which is not triply transitive.

Example. Let C be the set of all complex numbers and put $A = C \cup \{\infty\}$. Let G consist of all permutations π of A of the form $x\pi = \dfrac{ax+b}{cx+b}$ $(x \in A)$ where $a, b, c, d \in C$ with $|ad-bc|=1$. Then G is doubly transitive and the stabilizer G_∞ consists of all permutations π of the form $x\pi = ax+b$ where $a, b \in C$ and $|a|=1$. Thinking of the geometrical meaning of the permutations in G_∞, it is routine to show that G_∞ admits no nontrivial equivalence relation and nontrivial partial order on C.

References

[1] B. Csákány, Homogeneous algebras are functionally complete, *Algebra Universalis*, **11** (1980), 149—158.

[2] E. Fried, H. K. Kaiser and L. Márki, An elementary way for polynomial interpolation in universal algebras, *Algebra Universalis*, **15** (1982), 40—57.

[3] G. Grätzer, *Universal algebra*, 2nd ed., Birkhäuser Verlag, (Basel, 1979).

[4] H. K. Kaiser and L. Márki, Remarks on a paper of L. Szabó and Á. Szendrei, *Acta Sci. Math.*, **42** (1980), 95—98.

[5] R. A. Knoebel, Further conclusions on functional completeness, *Fund. Math.*, **99** (1978), 93—112.

[6] P. P. Pálfy, L. Szabó and Á. Szendrei, Algebras with doubly transitive automorphism groups, in: *Finite Algebra and Multiple-valued Logic* (Proc. Conf. Szeged, 1979), Colloq. Math. Soc. J. Bolyai, vol. 28, North-Holland (Amsterdam, 1981); pp. 321—335.

[7] P. P. Pálfy, L. Szabó and Á. Szendrei, Automorphism groups and functional completeness, *Algebra Universalis*, **15** (1982), 385—400.

[8] I. G. Rosenberg, Completeness properties of multiple-valued logic algebras, in: *Computer Science and Multiple-valued Logic. Theory and Applications* (ed. D. C. Rine), North Holland (Amsterdam, 1977).

[9] I. G. Rosenberg and L. Szabó, Local completeness, I, *Algebra Universalis*, to appear.

[10] A. A. Salomaa, A theorem concerning the composition of functions of several variables ranging over a finite set, *J. Symb. Logic*, **25** (1960), 203—208.

[11] A. A. Salomaa, On basic groups for the set of functions over a finite domain, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, **338** (1963), 1—15.

[12] P. Schofield, Complete subsets of mappings over a finite domain, *Proc. Cambridge Phil. Soc.*, **62** (1966), 597—611.

[13] R. Scott, *Group Theory*, Prentice-Hall, Inc. (Englewood Cliffs, N. J., 1964).

[14] L. Szabó, Interpolation in algebras with doubly primitive automorphism groups, *Elektron. Informationsverarb. und Kybernet.*, **19** (1983), 603—610.

[15] L. Szabó and Á. Szendrei, Almost all algebras with triply transitive automorphism groups are functionally complete, *Acta Sci. Math.*, **41** (1979), 391—402.

BOLYAI INSTITUTE
JÓZSEF ATTILA UNIVERSITY
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

Acta Sci. Math., 47 (1984), 71—83

Subalgebra lattices, simplicity and rigidity

VÁCLAV KOUBEK*

Let A be a universal algebra and let Con (A), Sub (A), End (A), Aut (A) denote the lattice of all congruences of A, the lattice of all subalgebras of A, the endomorphism monoid of A, and the automorphism group of A, respectively. It is well known that Con (A) and Sub (A) can be arbitrary algebraic lattices (see [5] and [2], or [4]), End (A) can be an arbitrary monoid (see [1]). W. A. LAMPE [8] proved the independence of Con (A), Sub (A) and Aut (A) — his construction represents each pair of non-trivial algebraic lattices and an arbitrary group as Sub (A), Con (A) and Aut (A) of a finitary algebra A. SAUER and STONE [9] characterize lattices L of subsets of a given set X and transformation monoids M on X for which there is an algebra A on X with $L=\mathrm{Sub}\,(A)$, $M=\mathrm{End}\,(A)$. This is an example of a concrete characterization, while the other results represent the lattices and/or groups up to an isomorphism of the respective structures.

The relationship between the lattice of subalgebras and automorphism groups of subalgebras was studied in [3] and the characterization for a special case was given. A concrete version of this problem was solved in [6]. The aim of this paper is to continue in these considerations — we characterize pairs (L, Δ) where L is an algebraic lattice and Δ is a finitary type such that there is an algebra A of type Δ with $\mathrm{Sub}\,(A)\cong L$ and each subalgebra of A (including A) is rigid and simple (an algebra A is *rigid* if End (A) is trivial, and it is *simple* if Con (A) is the two-element lattice).

The method of the presented proof is based on a construction given in [7]: for a given type Δ, it shows how large algebras of type Δ exist in which every subalgebra is rigid. We strenghten this result — we show that for a given non-unary type Δ and for a given cardinal α if there is an algebra A of type Δ with power $\geqq\alpha$ such that each of its subalgebras is rigid then there is an algebra B of type Δ with power $\geqq\alpha$ such that each of its subalgebras is rigid and simple.

Received October 6, 1982.

* The support of NSERC is gratefully acknowledged.

The basic algebraic notions can be found in [4], in particular, a type Δ is *finitary* if all operation symbols in Δ have finite arities, it is *non-unary (infinitary)* if there is an operation symbol in Δ with arity >1 (infinite arity). Denote by $|\Delta|$ the number of operation symbols in Δ. We add to the class Card of all cardinals a new largest element c (i.e. $c>\alpha$ for each cardinal α).

We recall some combinatorial notions given in [7]. For a cardinal α, a pair (X, φ) is called an *α-set pair* if X is a well-ordered set and φ is a mapping from the set of all finite subsets of X into a set Y with cardinality α. A one-to-one increasing sequence $\{x_0, x_1, \dots\}$ of elements of X (with respect to the well-ordering of X) is called *good* (w.r.t. (X, φ)) if for every finite subset A of $\{x_0, x_1, \dots\}$ and for every natural number n, $\varphi(A)=\varphi(\{x_{i+n};\ x_i\in A\})$. Let $\mathrm{Set}(\alpha)$ be the smallest cardinal such that each α-set pair (X, φ) with the power of X greater than or equal to $\mathrm{Set}(\alpha)$ has a good sequence; if such a cardinal does not exist put $\mathrm{Set}(\alpha)=c$. We prove:

Theorem 1. *If Δ is a finitary non-unary type, set $\alpha=\max\{2^{|\Delta|}, 2^{\aleph_0}\}$. The following are then equivalent for any cardinal β:*

(a) *β is smaller than* $\mathrm{Set}(\alpha)$;

(b) *there is an algebra (A, F) of type Δ such that each of its subalgebras is rigid and* $\mathrm{card}\, A\geqq\beta$;

(c) *there is an algebra (A, F) of type Δ such that each of its subalgebras is rigid and simple and* $\mathrm{card}\, A\geqq\beta$.

In [7] the equivalence of (a) with (b) was proved. Clearly, (c)$\Rightarrow$(b) and we are to prove (a)$\Rightarrow$(c). Moreover, as in [7] from this proof we obtain:

Corollary 2. *If Δ is an infinitary type then for each cardinal α there is an algebra (A, F) of type Δ such that each of its subalgebras is rigid and simple and* $\mathrm{card}\, A\geqq\alpha$.

For an algebraic lattice L put $c(L)=\sup\{\varkappa;$ there is a compact element c of L with $\varkappa=\mathrm{card}\{d\in L;\ d$ is compact, $d<c\}\}$, $d(L)=\min\{\alpha;\ \mathrm{Set}(\alpha)>\mathrm{card}\, L\}$ (In [7] it was proved that $\mathrm{Set}(\alpha)$ is a strongly inaccessible cardinal thus $\mathrm{Set}(\alpha)>\mathrm{card}\, L$ iff $\mathrm{Set}(\alpha)>\mathrm{card}\{c\in L;\ c$ is compact$\}$.) We prove the following:

Theorem 3. *For a given algebraic lattice L and for a given finitary non-unary type Δ there is an algebra A of type Δ such that*

(a) $\mathrm{Sub}(A)\cong L$;

(b) *each subalgebra of A is rigid and simple*

if and only if $2^\alpha>d(L)$, $\alpha\geqq c(L)$ where $\alpha=\max\{|\Delta|, \aleph_0\}$.

Let m be a regular cardinal. We recall that a complete lattice L is called m-algebraic (see [4]) if every $x\in L$ is a join of m-compact elements of L (an element x of L is m-compact if $x\leqq\bigvee M$ for a subset M of L only if there is a subset N of M

of cardinality less than m with $x \leqq \vee N$). An algebraic lattice is thus an $\aleph_0$-algebraic lattice. Put $c_m(L) = \sup \{\varkappa;$ there is an m-compact element c of L such that $\varkappa = \operatorname{card} \{d \in L;\ d$ is m-compact, $d < c\}\}$. For a type Δ denote $\sup \Delta = \sup \{\alpha^+;$ there is an operation symbol in Δ with the arity $\alpha\}$ (as usual, α^+ is the cardinal successor of α). Clearly, Δ is finitary iff $\sup \Delta \leqq \aleph_0$. It is well known that for each m-algebraic lattice L there are a type Δ with $\sup \Delta \leqq m$ and an algebra A of type Δ with $\operatorname{Sub}(A) \cong L$ (see [4]). We prove the following modification of Theorem 3:

Theorem 4. *Let m be a regular cardinal, $m > \aleph_0$. Then for a given m-algebraic lattice and for a given type Δ there is an algebra A of type Δ such that*

(a) $\operatorname{Sub}(A) \cong L$;

(b) *each subalgebra of A is rigid and simple*

if and only if $\sup \Delta \geqq m$ *and* $c_m(L) \leqq |\Delta| \cdot 2^{n\alpha}$ *for some* $n < m$, $\alpha < \sup \Delta$.

I wish to acknowledge helpful discussions with J. Sichler.

To prove the theorems we need some technical notions and claims. Let N denote the set of all natural numbers. If (A, F) is a partial algebra of type Δ then (A', F) is called a *subalgebra* of (A, F) if $A' \subset A$ and for $f \in F$, if $f(x_1, \ldots, x_n)$ is defined in (A, F) and $x_1, x_2, \ldots, x_n \in A'$ then $f(x_1, x_2, \ldots, x_n) \in A'$. An equivalence θ on A is called a *congruence* of (A, F) if for each $f \in F$ such that $f(x_1, x_2, \ldots, x_n)$, $f(y_1, y_2, \ldots, y_n)$ are defined and $x_i \theta y_i$ for each $i = 1, 2, \ldots, n$ we have $f(x_1, x_2, \ldots$ $\ldots, x_n)\, \theta\, f(y_1, y_2, \ldots, y_n)$. An algebra (A, G) of type Δ is called an *extension algebra* of (A, F) if the identity map is a homomorphism from (A, F) to (A, G).

A partial algebra (A, F) is called *strongly rigid (strongly simple)* if for every extension algebra (A, G) the identity map of A is the only homomorphism from (A, F) to (A, G) ((A, G) has only two congruences — trivial and total, respectively). A pair of partial algebras (A, F), (B, G) is called *strongly mutually rigid* if there is no homomorphism from (A, F), or (B, G), to any extension algebra of (B, G), or (A, F) respectively. The following proposition gives a reason for these definitions:

Proposition 5. *Let (A, F) be a partial algebra whose subalgebras are strongly rigid and strongly simple. Then each subalgebra of any extension algebra of (A, F) is rigid and simple.*

Proof is obvious.

The following lemma enables us to reduce the type Δ for which we shall prove Theorems 1 and 3.

Lemma 6. *Let $\Delta' = \{\eta_i;\ i \in I \cup \{0\}\}$ be a finitary type such that η_0 has arity greater than* 1, *let $\Delta = \{\sigma\} \cup \{\tau_i;\ i \in I\}$ be a type such that σ is binary and τ_i is nullary for every $i \in I$. If Theorem* 1 *or Theorem* 3 *holds for Δ then it also holds for Δ'.*

Proof. Since $\max\{2^{|\Delta|}, 2^{\aleph_0}\}=\max\{2^{|\Delta'|}, 2^{\aleph_0}\}$, (a) of Theorem 1 holds for Δ iff it holds for Δ'. Moreover, each algebra of type Δ can be represented as an algebra of type Δ' such that homomorphisms do not change. Hence we get the required assertions.

In what follows we assume that a type $\Delta=\{\sigma\}\cup\{\tau_i;\ i\in I\}$ is given such that σ is binary and τ_i is nullary for every $i\in I$. Denote by v the derived unary operation symbol, $v(x)=\sigma(x, x)$.

Choose a one-to-one mapping ζ from the set of all pairs (n, m) of positive integers, $n<m$, to $\mathbf{N}$ with $\zeta(n, m)>n+m$. Let $\{p_n;\ n\in\mathbf{N}\}$ be a one-to-one increasing sequence of positive integers such that $p_0=1$ and if $q=\zeta(n, m)$ then $p_q=p^k+p_{q-m+n}$ for some $k\in\mathbf{N}$, $k>0$ and a prime $p>m-n$. Lemma 8 below is a modification of a well known statement.

Construction 7. Let (A, F) be an algebra of type Δ such that $A=I\times\mathbf{N}$ and

(1) $$\tau_i(0) = (i, 0) \quad \text{for each} \quad i\in I,$$

(2) $$\sigma((i, 0), (i, n)) = (i, n+1) \quad \text{for each} \quad i\in I, \quad n\in\mathbf{N},$$

(3) $$\sigma((i, n), (i, n)) = (i, p_n) \quad \text{for each} \quad i\in I, \quad n\in\mathbf{N},$$

(4) $$\sigma((i, 2n), (i, 1)) = (i, 0) \quad \text{for each} \quad i\in I, \quad n\in\mathbf{N}, \quad n>0,$$

(5) $$\sigma((i, 2n+1), (i, 1)) = (i, 1) \quad \text{for each} \quad i\in I, \quad n\in\mathbf{N}, \quad n>0,$$

(6) $$\sigma((i, 0), (j, n)) = (i, n) \quad \text{for each} \quad i, j\in I, \quad i\neq j, \quad n\in\mathbf{N},$$

(7) $$\sigma((i, 1), (j, 0)) = (j, 0) \quad \text{for each} \quad i, j\in I, \quad i\neq j,$$

(8) the other values of σ are arbitrary.

Lemma 8. *The algebra (A, F) is simple, is generated by $\emptyset$,* $\operatorname{card} A=\max\{|\Delta|, \aleph_0\}$ *and $v^n(a)\neq a$ for each $a\in A$, $n\in\mathbf{N}$, $n>0$.*

Proof. Clearly, $\operatorname{card} A=\max\{|\Delta|, \aleph_0\}$. By (1) and (2), (A, F) is generated by $\emptyset$, by (2) and (3) $v^n(a)\neq a$ for each $a\in A$, $n\in\mathbf{N}$, $n>0$. Let θ be a non-trivial congruence on (A, F). If $(i, n)\,\theta\,(j, m)$ and $(i, n)\neq(j, m)$ then for $i\neq j$, $n\leq m$ get $(j, n)=\sigma((j, 0), (i, n))\,\theta\,\sigma((j, 0), (j, m))=(j, m+1)$. Therefore we may assume that for an $i\in I$, $(i, n)\,\theta\,(i, m)$, $n<m$, $n, m\in\mathbf{N}$. By (2) we get $(i, n+k)\,\theta\,(i, m+k)$ for each $k\in\mathbf{N}$, by (3), $(i, p_{n+k})\,\theta\,(i, p_{m+k})$. If $m+k=\zeta(n, m)$ then $m-n$ and $p_{m+k}-p_{n+k}=p^q$ are relatively prime because $p>m-n$ is a prime. If we combine these facts we see that $(i, q)\,\theta\,(i, q+1)$ for a sufficiently large $q\in\mathbf{N}$. If we use (2,) (4) and (5) we get $(i, 0)\,\theta\,(i, 1)$ and thus $(i, q)\,\theta\,(i, r)$ for each $q, r\in\mathbf{N}$ by (2). By (6), this holds for each $i\in I$ and from (7) we get $(i, 0)\,\theta\,(j, 0)$ for each $i, j\in I$. Thus $(i, q)\,\theta$ $\theta\,(j, r)$ for each quadruple $i, j\in I$, $q, r\in\mathbf{N}$; thus θ is total.

Construction 9. For every positive integer i a partial algebra (A_i, F_i) of type Δ will be constructed as follows: Let (A, F) be an algebra from Construction 7. Denote by A' a copy of the set A, for $a \in A$ we shall denote by a' the corresponding element in A'. Set $A_i = A \cup A' \cup \mathbf{N}$ (assume that A, A' and $\mathbf{N}$ are pairwise disjoint) and let F_i be an extension of F such that

(9) $\sigma(0, n) = n+1$ for each $n \in \mathbf{N}$,

(10) $\sigma(n+1, n) = 0$ for each $n \in \mathbf{N}$,

(11) $\sigma(n, n) = n+1$ for $n \in \mathbf{N}$, $n < i$, $\sigma(n, n) = 0$ for $n \geqq i$,

(12) $\sigma(a', a') = 1$ for each $a \in A$,

(13) $\sigma(0, a) = a'$ for each $a \in A$,

(14) $\sigma(0, a') = a$ for each $a \in A$,

(15) $\sigma(2, a) = 0$ for each $a \in A$,

(16) $\sigma(2, a') = 1$ for each $a \in A$,

(17) $\sigma(a', b') = 0$ for each $a, b \in A$, $a \neq b$,

(18) $\sigma(2, 2n) = 0$ for each $n \in \mathbf{N}$, $n > 2$,

(19) $\sigma(2, 2n+1) = 1$ for each $n \in \mathbf{N}$, $n > 2$,

(20) $\sigma(1, n) = p_n$ for each $n \in \mathbf{N}$, $n > 2$.

Lemma 10. *For each positive integer i, (A_i, F_i) satisfies:*

(a) *the operation ν is a total operation and it has exactly one cycle which has length $i+1$, this cycle is on the set $\{0, 1, \ldots, i\} = \nu(A' \cup \mathbf{N})$;*

(b) *(A_i, F_i) is strongly rigid;*

(c) *(A_i, F_i) is strongly simple;*

(d) *if $i \neq j \in \mathbf{N}$ then (A_i, F_i) and (A_j, F_j) are strongly mutually rigid;*

(e) *if (B, G) is a proper subalgebra of (A_i, F_i) then $(B, G) = (A, F)$;*

(f) *there are $\max\{|\Delta|, \aleph_0\}$ pairs $x, y \in A$ such that $\sigma(x, y)$ is not defined.*

Proof. (a) follows from (9), (11), (12) and from Lemma 8 ((A, F) is a total algebra). By a routine calculation from (9), (11), (12) and (13) we get (e). If $a, b, \in A$ then $\sigma(a, b')$ is not defined and (f) follows from Lemma 8. To verify (c), let (A_i, G) be an extension algebra of (A_i, F_i) and let θ be a non-trivial congruence on (A_i, G). The first task is to show that $0\theta 1$. The proof is divided into six steps:

(i) If $a, b \in A$, then $a\theta b$ iff $a'\theta b'$.

Indeed, this follows immediately from (13) and (14).

(ii) If $a, b \in A$, $a \neq b$, then $a' \theta b'$ implies $0\theta 1$.

Indeed, if we apply (12) and (17), then $a'\theta b'$ implies $0=\sigma(a', b')\,\theta\,\sigma(b', b')=1$.

(iii) If for $a \in A$, $n \in \mathrm{N}$, $a\theta n$, then $a'\,\theta\, n+1$.

By (9) and (13) we get $a'=\sigma(0, a)\,\theta\,\sigma(0, n)=n+1$.

(iv) If for $a \in A$, $n \in \mathrm{N}$ we have $a'\theta n$ then $0\,\theta\, 1$.

From (9), (13) and (14) we obtain $a'\,\theta\, n+2$; thus we can assume that $n>i$, in which case $1=\sigma(a', a')\,\theta\,\sigma(n, n)=0$.

(v) If $a, b, \in A$ such that $a\theta b'$ then $0\theta 1$.

$0=\sigma(2, a)\,\theta\,\sigma(2, b')=1$ follows by (15) and (16).

(vi) If $n, m \in \mathrm{N}$, $n \neq m$ then $n\theta m$ implies $0\theta 1$.

If $n\theta m$ then by (9) $n+k\,\theta\, m+k$ for each $k \in \mathrm{N}$, and if $2<n<m$ then $p_{n+k}\theta p_{m+k}$ for each $k \in \mathrm{N}$ by (20). If $k+m=\zeta(n, m)$ then $p_{m+k}-p_{n+k}=p^q$ for some $q \in \mathrm{N}$, $q \geqq 1$ and some prime $p>m-n$. Therefore $m-n$ and $p_{m+k}-p_{n+k}$ are relatively prime. Hence $q\,\theta\, q+1$ for some sufficiently large $q \in \mathrm{N}$, by a property of additive semigroups of natural numbers. Then by (18) and (19) we get $0\theta 1$.

Now we show that θ is total.

(vii) If $0\theta 1$ then $n\theta m$ for each $n, m \in \mathrm{N}$

follows immediately from (9).

(viii) If $0\theta 1$ then for each $a \in A$, $a\theta 0\theta a'$.

If $0\theta 1$ then $0\theta 2$ and by (13) and (15) $a'\theta 0$; by (14) and (16) $a\theta 1$.

Hence we conclude that (A_i, F_i) is strongly simple.

Next we prove (b). Let (A_i, G) be an extension algebra and $f\colon (A_i, F_i)\to(A_i, G)$ be a homomorphism. From (a) we see that $f(0)\in\{0, 1, \dots, i\}$. Hence $f(1)=f(\sigma(0, 0))=$ $=\sigma(f(0), f(0))=f(0)+1$ if $f(0)\neq i$, and $f(1)=0$ if $f(0)=i$. If $f(0)=i$ then $f(0)=$ $=f(\sigma(1, 0))=\sigma(f(1), f(0))=\sigma(0, f(0))=f(0)+1$, a contradiction. If $f(0)\neq i$ then $f(0)=f(\sigma(1, 0))=\sigma(f(1), f(0))=\sigma(f(0)+1, f(0))=0$. By (e) we obtain that $f=\mathrm{id}$ because $\{x; f(x)=x\}$ is a subalgebra of (A_i, F_i) containing 0.

To prove (d) consider an extension algebra (A_j, G) of (A_j, F_j) and a homomorphism $f\colon (A_i, F_i)\to(A_j, G)$. By (a) we get that $j\leqq i$; if $j<i$ then $\mathrm{Ker} f$ is a non-trivial congruence on (A_i, F_i), thus by (c) $\mathrm{Ker} f$ is the total congruence and f is constant. This contradicts (a).

Construction 11. Let $(X, \leqq)$ be a well-ordered set. We shall construct a partial algebra $(A(X), F(X))$ of type Δ. For every finite subset Y of X we take one copy of A, we denote it by $A(Y)$, and if $Y\neq\emptyset$ we take one copy of N, we denote it by $\mathrm{N}(Y)$. Set $A(X)=\bigcup\{A(Y); Y\subset X, Y$ is finite$\}\cup\bigcup\{\mathrm{N}(Y); Y\subset X, Y\neq\emptyset, Y$ is finite$\}$. For a finite $Y\subset X$ and for $x\in A\cup\mathrm{N}$, let $x(Y)$ be the corresponding point in $A(Y)\cup\mathrm{N}(Y)$. Further, define $\tau_Y\colon A_n\to A(X)$ (where $n=\mathrm{card}\, Y$) as follows: $\tau_Y(a)=a(\emptyset)$ for $a\in A$, $\tau_Y(a')=a(Y)$ for $a\in A$, $\tau_Y(n)=n(Y)$ for $n\in\mathrm{N}$. Denote by $F'(X)$ a set of partial operations on $A(X)$ of type Δ injectively generated by τ_Y, $Y\subset X$,

$\emptyset \neq Y$, Y is finite (i.e. $(A(X), F'(X))$ is the smallest partial algebra of type Δ such that $\tau_Y\colon (A_n, F_n) \to (A(X), F'(X))$ is a homomorphism for every finite non-empty subset Y of X). We shall identify $x \in X$ with $0(\{x\}) \in A(X)$, thus $X \subset A(X)$. For each pair of positive integers k, q choose a one-to-one mapping $\psi_{k,q}$ from the set of all pairs of natural numbers (n, m) with $n \leqq k, m \leqq q$ into $\mathbf{N}$ such that $\psi_{k,q} = \psi_{q,k}$. Let $(A(X), F(X))$ be the smallest extension (partial) algebra of $(A(X), F'(X))$ satisfying

(21) $\sigma(x, y) = \max\{x, y\}$ for each $x, y \in Y$, $x \neq y$,

(22) $\sigma(n(Y), m(Z)) = s(Y \cup Z)$ for each pair of distinct nonempty finite subsets Y, Z of X, for each pair $n, m \in \mathbf{N}$ with $n \leqq \operatorname{card} Y$, $m \leqq \operatorname{card} Z$ and either $n \neq 0$ or $m \neq 0$, where $s = \psi_{\operatorname{card} Y, \operatorname{card} Z}(n, m)$,

(23) $\sigma(3(Y), (4+k)(Y)) = y_k$ for each finite non-empty subset Y of X, for each $k < \operatorname{card} Y$, where y_k is the k-th point of Y.

By the choice of $\psi_{k,q}$ we get $\sigma(n(Y), m(Z)) = \sigma(m(Z), n(Y))$ in (22).

Proposition 12. *For every well-ordered set $(X, \leqq)$, $(A(X), F(X))$ fulfils:*

(a) *for every finite subset Y of X there are* $\max\{|\Delta|, \aleph_0\}$ *pairs x, y in A_n such that $\sigma(\tau_Y(x), \tau_Y(y))$ is not defined;*

(b) *for each subset Y of X, the subalgebra of $(A(X), F(X))$ generated by Y is on the set* $\bigcup\{\operatorname{Im} \tau_Z;\ \emptyset \neq Z \subset Y,\ Z$ *is finite*$\}$;

(c) *for every extension algebra $(A(X), G)$ of $(A(X), F(X))$, each subalgebra (B, G') of $(A(X), G)$ is generated by $B \cap X$;*

(d) *the lattice of subalgebras of $(A(X), F(X))$ is isomorphic to the lattice of all subsets of X;*

(e) *each subalgebra of $(A(X), F(X))$ is strongly simple;*

(f) *for every pair of finite subsets Y, Z of X with* $\operatorname{card} Y = \operatorname{card} Z$ *there is an isomorphism between the subalgebras of $(A(X), F(X))$ generated by Y and Z;*

(g) *for every extension algebra $(A(X), G)$ of $(A(X), F(X))$ and for every endomorphism φ of a subalgebra (B, G') of $(A(X), G)$ we have: φ is injective, $\varphi(B \cap X) \subset \subset B \cap X$, and the restriction of φ onto $B \cap X$ is increasing (i.e. if $x, y \in B \cap X$, $x < y$ then $\varphi(x) < \varphi(y)$);*

(h) *for every extension algebra $(A(X), G)$ of $(A(X), F(X))$, for every endomorphism φ of a subalgebra (B, G') of $(A(X), G)$, and for every finite subset Y of $B \cap X$ there is a unique isomorphism, namely φ, between the subalgebras of $(A(X), G)$ generated by Y and $\varphi(Y)$, respectively.*

Proof. A verification of (a) immediately follows from (f) of Lemma 10 and Construction 11. To prove (b) we note that if $x, y \in \operatorname{Im} \tau_Z$ for a finite set $Z \subset X$, then $\sigma(x, y) \in Z \cup \operatorname{Im} \tau_Z$ (by (23) and from the definition of $F'(X)$). By (22)

$\bigcup\{\mathrm{Im}\,\tau_Z;\ Z$ is a non-empty finite subset of $Y\}$ is a subalgebra of $(A(X), F(X))$. If we use (22) and (e) of Lemma 10 we see that this subalgebra is generated by Y.

To prove (c) we observe that if $z\in\mathrm{Im}\,\tau_Z$ for a finite subset Z of X then for each subalgebra (B, G) of an extension algebra $(A(X), G')$ of $(A(X), F(X))$ with $z\in B$ we have $Z\subset B$ (see (e) of Lemma 10 and (23)) and by (b) if $Z\subset B$ then $z\in B$. Condition (d) immediately follows from (b) and (c).

We prove (e). Let θ be a non-trivial congruence of an extension algebra (B, G) of a subalgebra (B, F') of $(A(X), F(X))$. We show that θ is total in the four steps that follow.

(i) If $a(\emptyset)\theta b(\emptyset)$ for some $a\neq b$, $a, b\in A$ then θ is total.
Lemma 8 implies that θ is total on $\mathrm{Im}\,\tau_\emptyset$, now by Lemma 10 θ is total on $\mathrm{Im}\,\tau_Y$ for each finite subset Y of $B\cap X$; (b) and (c) complete the proof.

(ii) If $a(\emptyset)\theta x(Y)$ for some $a\in A$, $x\in A\cup\mathrm{N}$, $\emptyset\neq Y\subset B\cap X$ then θ is total.
By (c) of Lemma 10 θ is trivial on $\mathrm{Im}\,\tau_Y$ and (i) concludes the proof.

(iii) If $n(Y)\,\theta\, m(Z)$ for $Y, Z\subset B\cap X$ with $0<\mathrm{card}\,Y<\mathrm{card}\,Z$, $n\leqq\mathrm{card}\,Y$, $m\leqq\mathrm{card}\,Z$ then θ is total.
If we use (a) of Lemma 10 we get that there are $p\neq q$, $p, q\in\mathrm{N}$ with $p(Y)\,\theta\, q(Y)$ (because $v^r(n(Y))\,\theta\, v^r(m(Z))$ for each $r\in\mathrm{N}$); now by (c) of Lemma 10 θ is total on $\mathrm{Im}\,\tau_Y$ and by (i) θ is total on B.

(iv) If $n(Y)\,\theta\, m(Z)$ for $Y, Z\subset B\cap X$ with $Z\neq Y$, $0<\mathrm{card}\,Y=\mathrm{card}\,Z$ and $n\leqq\mathrm{card}\,Y$, $m\leqq\mathrm{card}\,Z$ then θ is total.
Since $v^r(n(Y))\,\theta\, v^r(m(Z))$ for each $r\in\mathrm{N}$ we can assume that $0(Y)\,\theta\, k(Z)$ and $1(Y)\,\theta\, l(Z)$. If $l\neq 0$ then $\sigma(0(Y), l(Z))\ \theta\ \sigma(k(Z), 1(Y))$ and by (22) $\sigma(0(Y), l(Z))$, $\sigma(k(Z), 1(Y))$ are distinct elements of $\mathrm{Im}\,\tau_{Y\cup Z}$; thus θ is total on $\mathrm{Im}\,\tau_{Y\cup Z}$ by (c) of Lemma 10 and by (i) is total on B. If $l=0$ and $\mathrm{card}\,Y\geqq 2$ then $2(Y)\,\theta\, 1(Z)$ and by the same argument as above (we exchange pairs $1(Y)\,\theta\, 0(Z)$ and $2(Y)\,\theta\, 1(Z)$) we obtain that θ is total. If $l=0$ and $\mathrm{card}\,Y=1$ then $2(Y)=\sigma(0(Y), 1(Y))\,\theta\,\sigma(1(Z), 0(Z))=0(Z)\,\theta\, 1(Y)$ (we use that $\mathrm{card}\,Z=1$ implies $k=1$) hence θ is total on $\mathrm{Im}\,\tau_Y$, and θ is total by (i).

To finish the proof assume that $y(Y)\,\theta\, z(Z)$ where $y(Y)\neq z(Z)$. If $Y=Z$ then by (c) of Lemma 10 θ is total on $\mathrm{Im}\,\tau_Y$ and by (i) is total on B. If $Y\neq Z$ we note that $v^r(y(Y))\,\theta\, v^r(z(Z))$ for each $r\in\mathrm{N}$, and by (a) of Lemma 10 we obtain that the hypotheses of (iii) or (iv) are fulfilled. Thus θ is total and (B, G) is simple.

We prove (f). Let Y, Z be non-empty finite subsets of X with $\mathrm{card}\,Y=\mathrm{card}\,Z$. Then there is an order-preserving isomorphism $\psi\colon Y\to Z$ which maps the k-th element of Y onto the k-th element of Z. We shall extend ψ to an isomorphism of the subalgebras of $(A(X), F(X))$ generated by Y, Z respectively as follows: For a subset Y' of Y and for $x\in A\cup A'\cup\mathrm{N}$ we put $\psi(\tau_{Y'}(x))=\tau_{Z'}(x)$ where $Z'=\psi(Y')$. From (b) we see that ψ is a bijection of the subalgebra of $(A(X), F(X))$ generated by Y onto the subalgebra of $(A(X), F(X))$ generated by Z. A verification that it is a homomorphism is straightforward, we use (21), (22) and (23) and the definition of $F'(X)$. By the same way we get that the inverse of ψ is a homomorphism.

We prove (g). If φ is an endomorphism of a subalgebra (B, G') of an extension algebra $(A(X), G)$ of $(A(X), F(X))$ then by (e) it is one-to-one. For every $x \in B \cap X$ we have $\nu^2(x)=x$, thus $\nu^2(\varphi(x))=\varphi(x)$ and by Lemma 10 we get either $\varphi(x) \in X$ or $\varphi(x)=1(\{z\})$ for some $z \in X$. If $\varphi(x)=1(\{z\})$ then analogously as in Lemma 10, $\varphi(1(\{x\}))=z$ and $\varphi(x)=\varphi(\sigma(1(\{x\}), x))=\sigma(\varphi(1(\{x\})), \varphi(x))=\sigma(z, 1(\{z\}))=2(\{z\})$, a contradiction (we use (9), (10) and (11)). Therefore $\varphi(x) \in X$ and thus $\varphi(B \cap X) \subset B \cap X$. Now, (21) implies that the restriction of φ to $B \cap X$ is increasing.

Condition (h) is a consequence of (g) because, by (g), φ is uniquely determined on the generating set (see (c)).

Construction 13. Let (X, φ) be an α-set pair where $\alpha \leqq \max\{2^{|\Delta|}, 2^{\aleph_0}\}$. If Y is a non-empty finite subset of X and $a, b \in A$ then $\sigma(a(\emptyset), b(Y))$ is not defined in $(A(X), F(X))$ and if $n, m \in \mathrm{N}$, $n, m > 5$, $n \neq m+1$ then $\sigma(n(Y), m(Y))$ is not defined either. Choose a set $T \subset U = \{(a, b);\ a, b \in A\} \cup \{(n, m);\ n, m \in \mathrm{N},\ n, m > 5,\ n \neq m, m+1\}$ with card T=card $U-T=\max\{|\Delta|, \aleph_0\}$. Let $(A(X), F(X, \varphi))$ be an extension algebra of $(A(X), F(X))$ such that:

(24) For each non-empty finite subset Y of X and for each pair $(u, v) \in \bigcup \{\mathrm{Im}\, \tau_Z;\ Z \subset Y\}$, $\sigma(u, v)$ is defined if and only if $(u, v) \notin \{(a(\emptyset), b(Z));\ (a, b) \in T,\ a, b \in A,\ \emptyset \neq Z \subset Y\} \cup \{(n(Z), m(Z));\ (n, m) \in T,\ n, m \in \mathrm{N},\ \emptyset \neq Z \subset Y\}$ and $\sigma(u, v) \in \bigcup \{\mathrm{Im}\, \tau_Z;\ Z \subset Y\}$;

(25) if (Y, Z) is a pair of finite subsets of X with card Y=card Z and $\varphi(Y) \neq \varphi(Z)$ then there is a pair $(x, y) \in U-T$ such that for $x, y \in A$ if $z(Y)=\sigma(x(Y), y(\emptyset))$ then $z(Z) \neq \sigma(x(Z), y(\emptyset))$, for $x, y \in \mathrm{N}$ then $z(Y)=\sigma(x(Y), y(Y))$ implies $z(Z) \neq \sigma(x(Z), y(Z))$.

Evidently, since $\alpha \leqq \max\{2^{|\Delta|}, 2^{\aleph_0}\}$ and card $U=\max\{|\Delta|, \aleph_0\}$, conditions (24) and (25) can be fulfilled.

Proposition 14. *Let (X, φ) be an α-set pair without a good sequence and $\alpha \leqq \max\{2^{|\Delta|}, 2^{\aleph_0}\}$. Then $(A(X), F(X, \varphi))$ fulfils:*

(a) *each subalgebra of $(A(X), F(X, \varphi))$ is strongly simple and strongly rigid;*

(b) *the lattice of all subalgebras of $(A(X), F(X, \varphi))$ is isomorphic to the lattice of all subsets of X;*

(c) *for every finite subset Y of X there are* $\max\{|\Delta|, \aleph_0\}$ *pairs $x, y \in \mathrm{Im}\, \tau_Y$ such that $\sigma(x, y)$ is not defined in $(A(X), F(X, \varphi))$;*

(d) *if (B, G) is a finitely generated subalgebra of $(A(X), F(X, \varphi))$ then $B \cap X$ is finite and generates (B, G);*

(e) *if a subalgebra (B, G) of $(A(X), F(X, \varphi))$ is generated by $Y \subset X$ then $B \cap X = Y$.*

Proof. Since for $(a, b)\in T$, $a, b\in A$, $\sigma(a(\emptyset), b(Y))$ is not defined and for $(n, m)\in T$, $n, m\in \mathbf{N}$, $\sigma(n(Y), m(Y))$ is not defined and $\operatorname{card} T=\max\{|\Delta|, \aleph_0\}$, we get (c). Condition (b) immediately follows from (d) of Proposition 12 and (24). Conditions (d) and (e) follow from (b) and (c) of Proposition 12 and (24). By (e) of Proposition 12 and by the fact that $(A(X), F(X, \varphi))$ is an extension algebra of $(A(X), F(X))$ we get that each subalgebra of $(A(X), F(X, \varphi))$ is strongly simple. To show that it is strongly rigid, let (B, G) be an extension algebra of a subalgebra (B, F') of $(A(X), F(X, \varphi))$. Evidently, there is an extension algebra $(A(X), G')$ of $(A(X), F(X))$ such that (B, G) is a subalgebra of $(A(X), G')$. Thus if ψ is an endomorphism of (B, G) then ψ is injective, $\psi(B\cap X)\subset B\cap X$ and the restriction of ψ to $B\cap X$ is increasing by (g) of Proposition 12. Since $B\cap X$ is well-ordered, we have $\psi(x)\geqq x$ for each $x\in B\cap X$. If ψ is not identical then by (c) of Proposition 12 there is $x\in B\cap X$ with $\psi(x)\neq x$. Put $x_0=x$ and define $x_{i+1}=\psi(x_i)$. Then $x_0<x_1<<x_2<\dots$. We show that it is a good sequence with respect to (X, φ), which will be a contradiction. For a finite subset V of $\{x_0, x_1, \dots\}$, and for each $k\in\mathbf{N}$, $\psi^k(V)==\{x_{i+k};\ x_i\in V\}$. Then by (f) and (h) of Proposition 12 $\psi(\tau_V(x))=\tau_{\psi(V)}(x)$ for each $x\in A_n$ where $\operatorname{card} V=n$. Hence for each $(a, b)\in U-T$ if $\sigma(a(V), b(\emptyset))=x(V)$ and $a, b\in A$ then $\sigma(a(\psi(V)), b(\emptyset))=x(\psi(V))$, if $\sigma(a(V), b(V))=x(V)$ and $a, b\in\mathbf{N}$ then $\sigma(a(\psi(V)), b(\psi(V)))=x(\psi(V))$ and so $\varphi(V)=\varphi(\psi(V))==\varphi(\psi^k(V))$ for each $k\in\mathbf{N}$ (see (25)). Thus $\{x_0, x_1, \dots\}$ is a good sequence, a contradiction. Hence ψ is the identity and (a) holds.

Proof of Theorem 1. We are to show (a)⇒(c). By the definition of $\mathrm{Set}(\alpha)$, however, there is an α-set pair (X, φ) without a good sequence with $\operatorname{card} X\geqq\beta$. Furthermore, $\operatorname{card} A(X)\geqq\operatorname{card} X$; Lemma 5 and Proposition 14 now complete the proof.

Proof of Theorem 3. Let L be an algebraic lattice such that for $\alpha==\max\{|\Delta|, \aleph_0\}$, $2^\alpha>d(L)$, $\alpha\geqq c(L)$. Let C be the set of all compact elements of L. Since $2^\alpha>d(L)$ there is an 2^α-set pair (C, φ) without a good sequence. Consider a partial algebra $(A(C), F(C, \varphi))$ from Proposition 14. Let $(A(C), F(L))$ be an extension algebra of $(A(C), F(C, \varphi))$ such that:

(26) for each pair $c, d\in C$ with $c<d$ there is $(x, y)\in T$ such that if $x, y\in A$ then $\sigma(x(\emptyset), y(\{c\}))=d$, if $x, y\in\mathbf{N}$ then $\sigma(x(\{c\}), y(\{c\}))=d$;

(27) for each finite subset Y of C there is $(x, y)\in T$ such that if $x, y\in A$ then $\sigma(x(\emptyset), y(Y))=c=\vee Y$, if $x, y\in\mathbf{N}$ then $\sigma(x(Y), y(Y))=c=\vee Y$;

(28) for the other pairs $x, y\in A(C)$ such that $\sigma(x, y)$ is not defined, set $\sigma(x, y)=x$.

By (a) of Proposition 14 and by Proposition 5 each subalgebra is simple and rigid, moreover $(A(C), F(L))$ is a total algebra by (28). By (d) of Proposition 14 and by

(27) and (28) each finitely generated subalgebra of $(A(C), F(L))$ is generated by some $\{c\}$, $c\in C$ or by $\emptyset$. Moreover by (c) of Proposition 14 and by (26), (27) and (28) $c\leqq d$ in C iff the subalgebra of $(A(C), F(L))$ generated by $\{c\}$ is contained in the subalgebra of $(A(C), F(L))$ generated by $\{d\}$. By (e) of Proposition 14 the lattice of all subalgebras of $(A(C), F(L))$ is isomorphic to L. Conversely, if (A, F) is an algebra of type Δ such that each subalgebra of (A, F) is rigid and simple and the lattice of all subalgebras of (A, F) is isomorphic to L then by Theorem 1, $2^\alpha > d(L)$ where $\alpha = \max\{|\Delta|, \aleph_0\}$. Further, each finitely generated subalgebra (B, G) of (A, F) fulfils $\operatorname{card} B \leqq \alpha$ and thus it has at most α finitely generated subalgebras. Thus $c(L)\leqq\alpha$ and Theorem 3 is proved.

Proof of Corollary 2. Let Δ be an infinitary type. Analogously to Lemma 6 we see that in the proof of Corollary 2 we may assume that Δ contains one ω_0-ary operation ω. Let $\Delta' = \{\sigma\}$ be a type where σ is binary, let (X, φ) be an $\aleph_0$-set pair and let $(A(X), F(X, \varphi))$ be a partial algebra from Construction 13 of type Δ'. Define a partial algebra $(A(X), G(X, \varphi))$ of type Δ: choose a mapping η from the set of all increasing sequences in X onto $\{0, 1\}$ such that $\eta(\{x_0, x_1, \ldots\}) \neq \eta(\{x_1, x_2, \ldots\})$ for each increasing sequence $\{x_0 < x_1 < \ldots\}$ in X (see [7]),

(29) $$\omega(\{x_0, x_1, \ldots\}) = \sigma(x_0, x_1) \quad \text{if} \quad \{x_0, x_1, \ldots\}$$
is not one-to-one and $\sigma(x_0, x_1)$ is defined,

(30) $$\omega(\{x_0, x_1, \ldots\}) = x_{\eta(\{x_0, x_1, \ldots\})} \quad \text{if} \quad \{x_0, x_1, \ldots\}$$
is an increasing sequence of elements in X.

Then each subalgebra of $(A(X), G(X, \varphi))$ is strongly simple and strongly rigid. If (B, G) is an extension algebra of a subalgebra (B, G') of $(A(X), G(X, \varphi))$ then there is F' such that (B, F') is a subalgebra of $(A(X), F(X, \varphi))$ and each congruence θ on (B, G) is a congruence on (B, F'); hence (B, G) is simple. Each endomorphism ψ of (B, G) is also an endomorphism of (B, F') thus by (g) of Proposition 12 ψ is injective, $\psi(B\cap X)\subset B\cap X$ and ψ is increasing on $B\cap X$. Thus $\psi(x)\geqq x$ for each $x\in B\cap X$. If ψ is not identical, then by (c) of Proposition 12 there is an $x\in B\cap X$ with $\psi(x)\neq x$. Now we define an increasing sequence $x_0 = x$, $x_{i+1} = \psi(x_i)$ in $B\cap X$. By an easy calculation (see [7])

$$\psi(\omega(x_0, x_1, \ldots)) \neq \omega(\psi(x_0), \psi(x_1), \ldots) = \omega(x_1, x_2, \ldots),$$

a contradiction. Hence (B, G) is rigid. Now Proposition 5 concludes the proof.

Sketch of the proof of Theorem 4. Assume that L is an m-algebraic lattice and Δ is an infinitary type such that there is an algebra (A, F) of type Δ such that each of its subalgebras is simple and rigid, and the lattice of all subalgebras of (A, F) is isomorphic to L. It is well known that $m\leqq\sup\Delta$. Since m-compact elements correspond to subalgebras of (A, F) generated by a set of power less than m and because for each subalgebra (B, F') of (A, F) generated by a set of power less than m we have $\operatorname{card} B \leqq |\Delta| \cdot 2^{\alpha}$ where $\alpha < \sup \Delta$, we get that $c_m(L) \leqq |\Delta| \cdot 2^{n\alpha}$ for $n < m$, $\alpha < \sup \Delta$. Conversely, let L be an m-algebraic lattice and Δ be a type such that $m \leqq \sup \Delta$ and $c_m(L) \leqq |\Delta| \cdot 2^{n\alpha}$. We can assume that Δ contains a nullary operation symbol. It is routine to prove the following modification of Construction 7 and Lemma 8: *There is a simple algebra (A, F) of type Δ generated by $\emptyset$ such that* $\operatorname{card} A \geqq c_m(L)$ *and for an infinitary operation symbol $\sigma \in \Delta$ if we set $v(a) = \sigma(a, a, \ldots)$ then for each $a \in A$, $n \in \mathrm{N}$, $n \neq 0$, $v^n(a) \neq a$.* Now, if we substitute this algebra (A, F) into the construction of $(A(X), G(X, \varphi))$ described in the proof of Corollary 2, we obtain the existence of a partial algebra (B, G) of type Δ containing the set L' of all m-compact elements of L and such that:

(a) each subalgebra of (B, G) is strongly simple and strongly rigid;

(b) each subalgebra (C, G') of (B, G) is generated by $C \cap L'$;

(c) for each subset C of L', if (C', G') is a subalgebra of (B, G) generated by C then $C' \cap L' = C$;

(d) if a subalgebra (C, G') of (B, G) is generated by a set of power less than m then $\operatorname{card} C \cap L' < m$;

(e) for each subset C of L', $\operatorname{card} C < m$, if (C', G') is a subalgebra of (B, G) generated by C then there are at least $c_m(L)$ sequences τ, $\{x_i;\ i \in \operatorname{ar} \tau\}$ such that $\tau \in \Delta$, $x_i \in C'$ for $i \in \operatorname{ar} \tau$ and $\tau(\{x_i;\ i \in \operatorname{ar} \tau\})$ is not defined.

Arguments identical to those used in the proof of Theorem 3 show that there is an algebra of type Δ with the required properties.

We can observe that in the proof of Theorem 3 there are at least $\max\{|\Delta|, \aleph_0\}$ pairs such that $\sigma(x, y)$ is defined by (28), i.e. $\sigma(x, y) = x$. If we redefine a set of such pairs x, y such that $\sigma(x, y) = y$ then the required properties of algebra do not change — but two distinct such extension algebras are mutually rigid. Analogous consideration can be used for the proof of Theorem 4. We obtain:

Corollary 15. *Let m be a regular cardinal, L' the set of all m-compact elements of an m-algebraic lattice L. If Δ is a non-unary type such that $\sup \Delta \geqq m$ or $m = \aleph_0$ and $d(L) < \max\{2^{|\Delta|}, 2^{\aleph_0}\}$ and $c_m(L) \leqq |\Delta| \cdot \aleph_0^{n\alpha}$ where $n < m$, $\alpha < \sup \Delta$ in either case, then on a set of cardinality $\beta = \operatorname{card} L' \cdot |\Delta| \cdot \aleph_0^{\alpha}$ where $\alpha < \sup \Delta$ there are $\sup\{\beta^{\alpha};\ \alpha < \sup \Delta\}$ algebras of type Δ such that:*

(a) *each subalgebra of each algebra is rigid and simple;*

(b) *the lattice of all subalgebras of each algebra is isomorphic to L;*

(c) *each pair of algebras is mutually rigid.*

References

[1] M. AMBRUST, J. SCHMIDT, Zum Cayleyschen Darstellungssatz, *Math. Ann.*, **154** (1964), 70—72.

[2] G. BIRKHOFF, O. FRINK, Representations of lattices by sets, *Trans. Amer. Math. Soc.*, **64** (1948), 299—316.

[3] E. FRIED, J. SICHLER, On automorphism groups of subalgebras of universal algebra, *Acta Sci. Math.*, **40** (1978), 265—271.

[4] G. GRÄTZER, *Universal Algebra*, Second Edition, Springer-Verlag (Berlin—New York—Heidelberg, 1979).

[5] G. GRÄTZER, E. T. SCHMIDT, Characterization of congruence lattices of abstract algebras, *Acta Sci. Math.*, **24** (1963), 34—59.

[6] J. KOLLÁR, Automorphism groups of subalgebras; a concrete characterization, *Acta Sci. Math.*, **40** (1978), 291—295.

[7] V. KOUBEK, V. RÖDL, On combinatorial properties of universal algebras, to appear.

[8] W. A. LAMPE, The independence of certain related structures of a universal algebra, *Algebra Universalis*, **2** (1972), Part I, 99—112; Part II, 270—283; Part III, 286—295; Part IV, 296—302.

[9] N. SAUER, M. G. STONE, Endomorphism and subalgebra structure; a concrete characterization, *Acta Sci. Math.*, **39** (1977), 397—402.

COMPUTATIONAL CENTRE OF CHARLES UNIVERSITY
MALOSTRANSKÉ NÁM. 25
118 00 PRAHA 1, CZECHOSLOVAKIA

Acta Sci. Math., 47 (1984), 85—92

On additive functions satisfying a congruence

I. KÁTAI

1. Let f, g, u, v be real-valued completely additive functions,

$$L_n = f(n)+g(n+1)+u(n+2)+v(n+3).$$

We shall prove the following

Theorem. *If* $L_n \equiv 0 \pmod 1$ *for every* $n \geqq 1$, *then* f, g, u, v *assume integer values for every* n.

Corollary. *If* $L_n = 0$ *for every* $n \geqq 1$, *then* f, g, u, v *are indentically zero-functions.*

For the proof of the Corollary see [1].

Let A_N denote the assertion:

$$A_N: f(N),\ g(N),\ u(N),\ v(N) \equiv 0 \pmod 1.$$

Let $\mathcal{P}$ denote the set of primes. For the sake of brevity we shall put $a \equiv b$ instead of $a \equiv b \pmod 1$.

We shall prove our Theorem in two steps. First we shall prove Theorem 1′, after then Lemma 1:

Theorem 1′. *Theorem is true if* A_N *is true for* $N \leqq 11$.

Lemma 1. *If* $L_n \equiv 0 \pmod 1$ *for every* $n \geqq 1$, *then* A_N *is true for* $N \leqq 11$.

2. Proof of Theorem 1′. Assume that Theorem 1′ does not hold. Then there exists a smallest N for which A_N does not hold. From $L_{N-3} \equiv 0 \pmod 1$ it follows that $v(N) \equiv 0 \pmod 1$. Furthermore,

$$0 \equiv L_{N-2} \equiv u(N)+v(N+1) \pmod 1.$$

If $N+1 \notin \mathcal{P}$, then $v(N+1) \equiv 0 \pmod 1$, and so $u(N) \equiv 0 \pmod 1$. If $N+1 \in \mathcal{P}$, then N is even, and so $u(N) \equiv 0 \pmod 1$. Hence it follows that $f(N) \not\equiv 0 \pmod 1$,

Received October 6, 1982.

or $g(N)\not\equiv 0 \pmod 1$, and that $N\in\mathcal{P}$. Let $N=P\in\mathcal{P}$. Now we distinguish three cases:

(I) $f(P) \equiv \xi,\quad g(P) \equiv \eta,\quad \xi \not\equiv 0,\quad \eta \not\equiv 0;$

(II) $f(P) \equiv 0,\quad g(P) \equiv \eta,\quad \eta \not\equiv 0;$

(III) $f(P) \equiv \xi;\quad g(P) \equiv 0,\quad \xi \not\equiv 0.$

Lemma 2. (1) *Let* $3P+b=2Z$, $b\equiv 1 \pmod 3$, $Z+1<2P$. *Then* $u(Z)\equiv 0$, $v(Z)\equiv 0$.

(2) *Let* $3P+c=2U$, $c \equiv -1 \pmod 3$, $U+1<2P$. *Then* $f(U)\equiv 0$, $g(U)\equiv 0$.

Proof. (1) We may assume that $Z\in\mathcal{P}$. Since $Z\equiv -1 \pmod 3$, therefore all the prime factors occurring in $Z-3$, $Z-2$, $Z-1$, $Z+1$ are smaller than P. From $L_{Z-2}\equiv 0$, $L_{Z-3}\equiv 0$ we get that $u(Z)\equiv 0$, $v(Z)\equiv 0$.

(2) We may assume that $U\in\mathcal{P}$. Since $U\equiv 1 \pmod 3$, therefore all the prime factors occurring in $U-1, U+1, U+2, U+3$ are smaller than P. From $L_U\equiv 0$, $L_{U-1}\equiv 0$ we get that $f(U)\equiv 0$, $g(U)\equiv 0$.

Case (I). Observing that $P-1, P+1, P+3$ are even numbers with prime factors $<P$, we get from $L_{P-1}\equiv 0$, $L_P\equiv 0$ that $U(P+2)\equiv -\xi$, $g(P+2)\equiv -\xi$, and so $P+2\in\mathcal{P}$, $\mathcal{P}\equiv -1 \pmod 3$. Similarly, in view of $2P+5\equiv 0 \pmod 3$, $2P+5\leqq 3P$, we see that $g(2P+3)\equiv -\xi$, $2P+3\in\mathcal{P}$. Since $2P-1\equiv 0 \pmod 3$, $2 \mid P+1$, therefore $L_{2P-1}\equiv 0$ implies that $u(2P+1)\equiv -\xi$, $2P+1\in\mathcal{P}$.

Now we shall prove that $3P+2\not\equiv 0 \pmod 7$, i.e., $P\not\equiv 4 \pmod 7$. Indeed, if $7 \mid 3P+2$, then from $L_{3P-1}\equiv 0$ we infer that

$$0 \equiv f(3P-1)+g(3P)+u(3P+1),$$

which gives that $f(3P-1)\not\equiv 0$ or $u(3P+1)\equiv 0$, but this is impossible as it was proved in Lemma 2. Since $P\not\equiv 4 \pmod 7$, and P, $P+2, 2P+1, 2P+3\in\mathcal{P}$, we get that $P\not\equiv 0, 2, 3, 4, 5 \pmod 7$; consequently $P\equiv 1$ or $6 \pmod 7$.

First, by considering $L_{2P-2}\equiv 0$ we deduce that $v(2P+1)\equiv 0$, and hence, by $L_{4P-1}\equiv 0 \pmod 5$, and by taking into account that $5 \mid 4P-1$ we get that $g(4P)+u(4P+1)\equiv 0$, i.e., $u(4P+1)\equiv -\xi$. So $4P+1=3R$, $u(R)\equiv -\eta$. It is obvious that $R\in\mathcal{P}$, since in the opposite case all its prime factors would be smaller than P. From $L_{R-2}\equiv 0$, by observing that $(R+1)/2<P$, we deduce that $f(R-2)\equiv\eta$, and so that $R-2\in\mathcal{P}$. Since $3(R-2)=4P-5$, therefore $f(4P-5)\equiv\eta$, and so

$$0 \equiv \eta+g(4P-4)+u(4P-3)+v(4P-2) \equiv \eta+u(4P-3).$$

Since $2\nmid 4P-3$, $3\nmid 4P-3$, therefore $P\not\equiv 6 \pmod 7$.

It remains to consider the case $P\equiv 1 \pmod 7$. Then $3R\equiv 5 \pmod 7$, $R\equiv 4 \pmod 7$, $2R-1\equiv 0 \pmod 7$. Let us consider now

$$0 \equiv L_{2R-2} \equiv f(2R-2)+g(2R-1)+u(2R)+v(2R+1).$$

Since $R, R-2 \in \mathcal{P}$, therefore $R \equiv 1 \pmod 3$, consequently $3 \mid 2R+1$. Furthermore $(2R+1)/3 < P+2$, and so $v(2R+1) \equiv 0$. Since $4 \mid 2R-2$, $7 \mid 2R-1$, therefore $f(2R-2) \equiv 0$, $g(2R-1) \equiv 0$, whence $u(R) \equiv 0$, which contradicts $u(R) \equiv -\eta$. So we have proved that Case (I) cannot occur.

Case (II). We get as earlier that $v(P+2) \equiv -\eta$, $P+2 \in \mathcal{P}$, and so $P \equiv -1 \pmod 3$. Since $3 \mid 2P+1$, therefore from $L_{2P-1} \equiv 0$ we infer that $u(2P+1) \equiv -\eta$, $2P+1 \in \mathcal{P}$. Lemma 2 implies that $f(3P-1) \equiv 0$, $u(3P+1) \equiv 0$, and so from $L_{3P-1} \equiv 0$ we deduce that $v(3P+2) \equiv -\eta$, $3P+2 \in \mathcal{P}$. Since P, $P+2$, $2P+1$, $3P+2 \in \mathcal{P}$, therefore $P \equiv -1 \pmod 5$. From $L_{2P-2} \equiv 0$ it follows that $v(2P+1) \equiv 0$, and so by $L_{4P-1} \equiv 0$, $5 \mid 4P-1$, we have

$$0 \equiv L_{4P-1} \equiv f(4P-1) + g(4P) + u(4P+1) + v(4P+2) \equiv 0 + \eta + u(4P+1) + 0,$$

hence $u(4P+1) \equiv -\eta$.

Thus $4P+1 = 3R$, $u(R) \equiv -\eta$, and so $R \in \mathcal{P}$. From $L_{R-2} \equiv 0$ we deduce that $f(R-2) \equiv \eta$, $R-2 \in \mathcal{P}$. Consequently $R \equiv 1 \pmod 3$. Now we have $f(4P-5) = f(3(R-2)) \equiv \eta$, implying

$$0 \equiv L_{4P-5} = f(4P-5) + g(4P-4) + u(4P-3) + v(4P-2) \equiv \equiv \eta + 0 + u(4P-3) + v(2P-1). \tag{2.1}$$

Now we shall prove that $v(2P-1) \equiv 0 \pmod 1$. Indeed,

$$0 \equiv L_{2P-4} \equiv f(2P-4) + g(2P-3) + u(2P-2) + v(2P-1),$$

whence by $5 \mid 2P-3$ it follows immediately that $v(2P-1) \equiv 0$, and so from (2.1), $u(4P-3) \equiv -\eta$, $4P-3 \in \mathcal{P}$. Since P, $P+2$, $2P+1$, $3P+2$, $4P-3$, $4P+3 \in \mathcal{P}$, therefore $P \equiv 2 \pmod 7$. From $4P+1 = 3R$, $R \equiv 1 \pmod 8$ we get that $P \equiv 5 \pmod 9$.

Let us consider now the relation

$$0 \equiv f(5P-1) + g(5P) + u(5P+1) + v(5P+2).$$

We have $7 \mid 5P+2$, $6 \mid 5P-1$, and so $f(5P-1) \equiv 0$, $v(5P+2) \equiv 0$, yielding $u(5P+1) \equiv -\eta$. Thus $5P+1 = 4X$ or $5P+1 = 2X$ with a prime $X > P$. First we consider the case $5P+1 = 4X$. Since $u(X) \not\equiv 0$, therefore from $L_{X-2} \equiv 0$ we get that $f(X-2) \not\equiv 0$. But, from $P \equiv -1 \pmod 3$ we get that $X \equiv -1 \pmod 3$, $3 \mid X-2$, $(X-2)/3 < P$, where $f(X-2) \equiv 0$.

It remains to consider the case $5P+1 = 2X$, $X \in \mathcal{P}$. We have $u(X) \equiv -\eta$. Furthermore $X \equiv 2 \pmod 7$. So

$$0 \equiv L_{X-2} \equiv f(X-2) + g(X-1) + u(X) + v(X+1).$$

Observing that $7 \mid X-2$, $6 \mid X-1$, and that $X/6 < P$, we get that

$$v(X+1) \equiv \eta. \tag{2.2}$$

Taking into account that $2X+2=5P+3$, $9|5P+2$, from $L_{5P}\equiv 0$ we deduce that $g(5P+1)\equiv -\eta$, i.e., $g(X)\equiv -\eta$. This, together with

$$0\equiv f(X-1)+g(X)+u(X+1)+v(X+2),$$

$3 \mid X+2$, and $(X+2)/3<P$, implies that

$$u(X+1)\equiv \eta. \tag{2.3}$$

Consequently $X+1=2Z$.

From (2.2) and (2.3) we get that $u(Z)\equiv\eta$, $v(Z)\equiv\eta$, $Z\in\mathcal{P}$. Using $(Z+1)/2<P$ and $2 \mid Z+1, Z-1, Z-3$, we see immediately that $f(Z-2)\equiv -\eta$, $g(Z-2)\equiv -\eta$, $Z-2\in\mathcal{P}$. Since $2(Z-2)=X-3$, we have $g(X-3)\equiv -\eta$. Let us consider the relation

$$0\equiv f(X-4)+g(X-3)+u(X-2)+v(X-1).$$

In view of $X\equiv 2$ (7), $u(X-2)\equiv 0$. Furthermore $2, 3 \mid X-1$, and so $v(X-1)\equiv 0$. Consequently $f(X-4)\equiv\eta$. But this is impossible, since $3 \mid X-4$, $(X-4)/3<P$.

Case (III). From $L_P\equiv 0$ we get that $u(P+2)\equiv -\xi$, $P+2\in\mathcal{P}$. Hence $P\equiv -1$ (mod 3). Observing that $3 \mid 2P+5$, we get from $L_{2P+2}\equiv 0$, that

$$g(2P+3)\equiv \xi, \quad 2P+3\in\mathcal{P}. \tag{2.4}$$

Let us consider now the relation

$$f(3P+4)+g(3P+5)+u(3P+6)+v(3P+7)\equiv 0.$$

From Lemma 2 we get that $g(3P+5)\equiv 0$, $v(3P+7)\equiv 0$, thus $f(3P+4)\equiv\xi$, $3P+4\in\mathcal{P}$. Since, $P, P+2, 2P+3, 3P+4\in\mathcal{P}$, therefore $P\equiv -1$ (mod 5).

Furthermore $L_{2P+3}\equiv 0$ immediately implies that $f(2P+3)\equiv 0$. Thus, by $5 \mid 4P+9$, we get that

$$0\equiv L_{4P+6}\equiv f(4P+6)+g(4P+7)+u(4P+8)+v(4P+9)\equiv$$
$$\equiv 0+g(4P+7)+u(P+2)+0,$$

i.e., $g(4P+7)\equiv\xi$.

Let $4P+7=3E$, $g(E)\equiv\xi$, $E\in\mathcal{P}$. From $L_{E-1}\equiv 0$ we deduce that $v(E+2)\equiv -\xi$. Hence it follows that $E\equiv -1$ (mod 3) and so $P\equiv 2$ (mod 9). Now we prove that $u(E)\equiv 0$. Indeed, in the opposite case from $L_{E-2}\equiv 0$ it would follow that $f(E-2)\not\equiv 0$, but this is impossible since $3 \mid E-2$, $(E-2)/3<P$.

So we have that $u(3E)\equiv u(4P+7)\equiv 0$. Then

$$0\equiv f(4P+5)+g(4P+6)+u(4P+7)+v(4P+8)\equiv$$
$$\equiv f(4P+5)+g(2P+3)+0+v(P+2).$$

From $L_{P-1}\equiv 0$ we get that $g(P)+v(P+2)\equiv 0$, and so $v(P+2)\equiv 0$. Using (2.4) we see that $f(4P+5)\equiv -\xi$, $4P+5\in\mathcal{P}$. Since $P\equiv -1$ (mod 5), we get that $E\equiv 1$

(mod 5). Consequently $5 \mid 2E+3$. So

$$0 \equiv f(2E+1)+g(2E+2)+u(2E+3)+v(2E+4) \equiv f(2E+1)+0+0-\xi,$$

i.e., $f(2E+1)\equiv\xi$, $2E+1\in\mathscr{P}$. Similarly, $3 \mid 2E-1$, therefore

$$0 \equiv f(2E-1)+g(2E)+u(2E+1)+v(2E+2) \equiv 0+\xi+u(2E+1)+0,$$

i.e., $u(2E+1)\equiv-\xi$. We have $3(2E+1)=8P+17$, hence

$$0 \equiv f(8P+15)+g(8P+16)+u(8P+17)+v(8P+18) \equiv$$
$$\equiv f(8P+15)+g(P+2)-\xi+v(4P+9).$$

Since $4 \mid P+4$, we get from $L_{P-1}\equiv 0$ that $g(P+2)\equiv 0$. Also, $5 \mid 4P+9$ implies that $v(4P+9)\equiv 0$. Thus we have that $f(8P+15)\equiv\xi$.

Hence $8P+15$ has to be a prime or the product of 7 and K, where $K\in\mathscr{P}$, $f(K)\equiv\xi$. Assume that $8P+15=7K, f(K)\equiv\xi$. Then we get from $L_K\equiv 0$ that $u(K+2)\equiv-\xi$, $K+2\in\mathscr{P}$. But $8P+15\equiv 7K$, $P\equiv-1 \pmod 3$ imply that $3 \mid K+2$, and hence, by $(K+2)/2<P$, $u(K+2)\equiv 0$.

So $8P+15\in\mathscr{P}$. Since $P, P+2, 2P+3, 3P+4, 4P+5, 8\mathscr{P}+15\in\mathscr{P}$, therefore $P\equiv 3 \pmod 7$. Let us consider now the relation

$$0 \equiv f(5P+8)+g(5P+9)+u(5P+10)+v(5P+11).$$

Since $9 \mid 5P+8$, $6 \mid 5P+11$, and $u(5P+10)\equiv u(P+2)\equiv-\xi$, therefore $f(5P+8)\equiv 0$, $v(5P+11)\equiv 0$, and so $g(5P+9)\equiv\xi$. Then $5P+9=2A$, or $5P+9=4A$, where $A\in\mathscr{P}$, $g(A)\equiv\xi$. The second case cannot occur. Let us assume that $5P+9=4A$, $g(A)\equiv\xi$. Then, taking into account that $2 \mid A-1$, $A+1$, $(A+1)/2<P$, we get from $L_{A-1}\equiv 0$ that $v(A+2)\equiv-\xi$. But this is impossible since $3 \mid A+2$.

Let us assume that $5P+9=2A$. It follows from $P\equiv 3 \pmod 7$ that $A\equiv 5 \pmod 7$, i.e., $7 \mid A+2$. Furthermore, $3 \mid A+1$, $(A+1)/3<P$, consequently $u(A+1)\equiv 0$, $v(A+2)\equiv 0$, and so $L_{A-1}\equiv 0$ immediately implies that $f(A-1)\equiv-\xi$. Since $A-1$ is an even number and has a prime divisor greater than P, therefore $A-1=2B$, $B\in\mathscr{P}$, $f(B)\equiv-\xi$. From $L_B\equiv 0$ we deduce that $u(B+2)\equiv\xi$. Since $5P+7=4B$, $9 \mid 5P+8$, $v(P+2)\equiv 0$, we get

$$0 \equiv f(5P+7)+g(5P+8)+u(5P+9)+v(5P+10) \equiv -\xi+u(2A),$$

i.e., $u(A)\equiv\xi$. So we have

$$f(A-2)+g(A-1)+u(A)+v(A+1) \equiv 0.$$

Since $3 \mid A-2, A+1$, and $(A+1)/3<P$, therefore $f(A-2)\equiv 0$, $v(A+1)\equiv 0$, and so $g(A-1)\equiv g(B)\equiv-\xi$. In view of $L_{B-1}\equiv 0$ this yields that

$$v(B+2) \equiv \xi. \tag{2.5}$$

Since $2B+4=A+3$, we have that $u(A+3)\equiv\xi$, $v(A+3)\equiv\xi$. Let us consider now the relation

$$f(A+1)+g(A+2)+u(A+3)+v(A+4)\equiv 0.$$

Since $7\mid A+2$, therefore $g(A+2)\equiv 0$. Furthermore $3\mid A+1$, $3\mid A+4$. As

$$(A+1)/3=(2A+2)/6=(5P+11)/6<P$$

for $P>11$, we have $f(A+1)\equiv 0$. We know that $v(P)\equiv 0$ and $v(P+2)\equiv 0$. Since $P, P+2\in\mathcal{P}$, therefore $P+4$ is a composite number, and so the smallest integer on which v assumes a nonzero value (mod 1) is $\geqq P+6$. However,

$$(A+4)/3=(2A+8)/6=(5P+17)/6<P+6,$$

therefore $v(A+4)\equiv 0$. Consequently $u(A+3)\equiv 0$, contradicting (2.5).

The proof of Theorem 1′ is finished.

3. Proof of Lemma 1. For an arbitrary completely additive function $h(n)$ we can extend the domain of definition for the set of positive rational numbers by $h(a/b)=h(a)-h(b)$. Let us do it for f, g, u, v. For the sake of brevity the relation

$$f(a)+g(b)+u(c)+v(d)\equiv 0 \pmod 1$$

will be denoted by $\langle a, b, c, d\rangle$, where a, b, c, d are arbitrary positive rational numbers.

From the additivity it follows that

$$\text{if}\quad \langle a,b,c,d\rangle \quad\text{and}\quad \langle A,B,C,D\rangle,\quad \text{then}\quad \langle aA, bB, cC, dD\rangle.$$

We shall say that $\langle aA, bB, cC, dD\rangle$ is the product of $\langle a, b, c, d\rangle$ and $\langle A, B, C, D\rangle$. It is obvious that $\langle 1/a, 1/b, 1/c, 1/d\rangle$ holds if $\langle a, b, c, d\rangle$ holds.

Let now $L_n=\langle n, n+1, n+2, n+3\rangle$. First we shall express the values $f(p), g(p), u(p), v(p)$ for primes $p\leqq 20$ as linear combinations of

$$K=\{f(2),\ g(2),\ u(2),\ v(2),\ f(3),\ g(3),\ u(3),\ v(3)\}.$$

The appropriate formulas will be denoted by $F(p), G(p), U(p), V(p)$. Hence we can get some linear relations between the values listed in K.

$$V(5)=L_2=\langle 2;\ 3;\ 2^2;\ 5\rangle,$$

$$U(5)=L_3=\langle 3;\ 2^2;\ 5;\ 2\cdot 3\rangle,$$

$$F(7)=L_7L_2^{-1}=\langle 7\cdot 2^{-1};\ 2^3\cdot 3^{-1};\ 2^{-2}\cdot 3^2;\ 2\rangle,$$

$$G(7)=L_6=\langle 2\cdot 3;\ 7;\ 2^3;\ 3^2\rangle,$$

$$V(11)=L_8L_3^{-1}=\langle 2^3\cdot 3^{-1};\ 2^{-2}\cdot 3^2;\ 2;\ 11\cdot 2^{-1}\cdot 3^{-1}\rangle,$$

$$V(17)=L_{48}L_3^{-2}L_6^{-2}=\langle 2^2\cdot 3^{-3};\ 2^{-4};\ 2^{-5};\ 17\cdot 2^{-2}\cdot 3^{-5}\rangle,$$

$$G(5)=L_{14}(F(7)V(17))^{-1}=\langle 3^3;\ 2\cdot 3^2\cdot 5;\ 2^{11}\cdot 3^{-2};\ 2\cdot 3^5\rangle,$$

$$V(7) = L_4 G(5)^{-1} = \langle 2^2\cdot 3^{-3};\ 2^{-1}\cdot 3^{-2};\ 2^{-10}\cdot 3^3;\ 7\cdot 2^{-1}\cdot 3^{-5}\rangle,$$

$$F(11) = L_{24}L_4L_{11}^{-1}G(5)^{-3} = \langle 11^{-1}\cdot 2^5\cdot 3^{-8};\ 2^{-5}\cdot 3^{-7};\ 2^{-31}\cdot 3^7;\ 2^{-4}\cdot 3^{-12}\rangle,$$

$$F(5^3) = L_5L_{25}V(5)L_{12}^{-1}V(7)^{-1} = \langle 5^3\cdot 2^{-3}\cdot 3^2;\ 2^3\cdot 3^4;\ 2^{11};\ 2^6\cdot 3^4\rangle,$$

$$V(13) = \frac{L_{54}L_{15}L_{12}^2L_7L_3F(5^3)}{L_{168}L_{10}L_5^3L_2^3G(5)} = \langle 2^{-5}\cdot 3^5;\ 2^8\cdot 3^{-4};\ 2^{-4}\cdot 3^3;\ 13^{-1}\cdot 2^{-1}\cdot 3^3\rangle,$$

$$F(5) = L_{75}L_4L_3V(13)L_5^{-1}L_{18}^{-1}L_9^{-1} = \langle 5\cdot 2^{-4}\cdot 3^3;\ 2^{10}\cdot 3^{-5};\ 2^{-5}\cdot 3^4;\ 2^{-4}\cdot 3^3\rangle,$$

$$U(7) = F(5)L_5^{-1} = \langle 2^{-4}\cdot 3^3;\ 2^9\cdot 3^{-6};\ 7^{-1}\cdot 2^{-5}\cdot 3^4;\ 2^{-7}\cdot 3^3\rangle,$$

$$G(13) = L_{12}U(7)L_2^{-1} = \langle 2^{-3}\cdot 3^4;\ 13\cdot 2^9\cdot 3^7;\ 2^{-6}\cdot 3^4;\ 2^{-7}\cdot 3^4\rangle,$$

$$G(11) = F(5)V(13)^{-1}L_{10}^{-1} = \langle 3^{-4};\ 11^{-1}\cdot 2^2\cdot 3^{-1};\ 2^{-3};\ 2^{-3}\rangle,$$

$$U(17) = F(5)L_{15}^{-1} = \langle 2^{-4}\cdot 3^2;\ 2^6\cdot 3^{-5};\ 17^{-1}\cdot 2^{-5}\cdot 3^4;\ 2^{-4}\cdot 3\rangle,$$

$$U(13) = F(11)L_{11}V(7)^{-1} = \langle 2^3\cdot 3^{-5};\ 2^{-2}\cdot 3^{-4};\ 13\cdot 2^{-21}\cdot 3^4;\ 2^{-2}\cdot 3^{-7}\rangle,$$

$$U(11) = L_9G(5)^{-1} = \langle 3^{-1};\ 3^{-2};\ 11\cdot 2^{-11}\cdot 3^2;\ 2\cdot 3^{-4}\rangle,$$

$$V(19) = L_{54}G(11)U(7)G(5)^{-1} = \langle 2^{-3}\cdot 3^{-1};\ 2^{10}\cdot 3^{-9};\ 2^{-16}\cdot 3^6;\ 19\cdot 2^{-11}\cdot 3^{-1}\rangle,$$

$$G(17) = V19)L_{16}^{-1} = \langle 2^{-7}\cdot 3^{-1};\ 17^{-1}\cdot 2^{10}\cdot 3^{-9};\ 2^{-17}\cdot 3^4;\ 2^{-11}\cdot 3^{-1}\rangle,$$

$$F(17) = \frac{L_{85}L_9L_6L_7L_2}{L_{42}L_{27}F(5)G(5)V(11)} = \langle 17\cdot 2^2\cdot 3^{-6};\ 3^2\cdot 2^{-6};\ 3\cdot 2^{-4};\ 2^9\cdot 3^{-7}\rangle,$$

$$U(19) = F(17)L_2L_{17}^{-1} = \langle 2^3\cdot 3^{-6};\ 2^{-7}\cdot 3;\ 19^{-1}\cdot 2^{-2}\cdot 3;\ 2^7\cdot 3^{-7}\rangle,$$

$$G(19) = L_{18}V(7)^{-1}L_3^{-2} = \langle 2^{-1}\cdot 3^3;\ 19\cdot 2^{-3}\cdot 3^2;\ 2^{11}\cdot 3^{-3};\ 2^{-1}\cdot 3^4\rangle,$$

$$F(19) = L_{19}U(7)V(11)^{-1}G(5)^{-1} = \langle 19\cdot 2^{-5}\cdot 3;\ 2^{12}\cdot 3^{-10};\ 2^{-17}\cdot 3^7;\ 2^{-6}\cdot 3^{-1}\rangle.$$

$$F_1 := L_1 = \langle 1;\ 2;\ 3;\ 2^2\rangle,$$

$$F_2 := \frac{F(5)^3L_1^{12}}{F(5)^3} = \left\langle \frac{2^9}{3^7};\ \frac{3^{19}}{2^{15}};\ 2^{26};\ \frac{2^{42}}{3^5}\right\rangle,$$

$$F_3 := \frac{L_{340}L_2L_8G(11)L_1^7}{L_{17}L_{30}L_3V(7)^3} = \left\langle \frac{3^2}{2};\ 2^9\cdot 3^6;\ 2^{25};\ 2^{11}\cdot 3^{13}\right\rangle,$$

$$F_4 := \frac{L_{133}L_{20}L_{15}L_2G(5)^2L_1^{21}}{L_{66}L_{14}L_9L_3L_7L_6F(19)F(11)F(5)} = \left\langle 2^6;\ 2^7\cdot 3^{27};\ 2^{69};\ \frac{2^{58}}{3^{17}}\right\rangle,$$

$$F_5 := \frac{L_{32}G(11)U(17)}{L_2L_1V(7)} = \left\langle \frac{3}{2^2};\ \frac{2^8}{3^4};\ 2;\ \frac{3^6}{2^9}\right\rangle,$$

$$F_6 := \frac{L_3L_1^{15}}{L_{33}F(11)G(17)U(7)} = \langle 2^6\cdot 3^6;\ 2^2\cdot 3^{22};\ 2^{53};\ 2^{63}\cdot 3^9\rangle,$$

$$F_7 := \frac{L_{31}L_6L_4L_2L_1^8}{L_{62}L_9} = \langle 2^4;\ 2^5\cdot 3^5;\ 2^{19};\ 2^7\cdot 3^8\rangle,$$

$$F_8 := \frac{L_7^2G(5)^2L_1^8}{L_{49}L_2^2U(17)V(13)} = \left\langle \frac{2^7}{3};\ 2\cdot 3^{11};\ 2^{27};\ 2^{24}\cdot 3^6\right\rangle,$$

$$F_9 := \frac{L_{63}L_2L_1^6}{L_8L_7U(13)} = \left\langle \frac{3^7}{2^5};\ 2^{11}\cdot 3^3;\ 2^{22};\ 2^{14}\cdot 3^8\right\rangle,$$

$$F_{10} := \frac{L_{55}L_{13}F(11)U(19)}{L_{26}L_6^3L_3U(7)F(5)L_1^2} = \left\langle \frac{2^{13}}{3^{23}};\ \frac{3^2}{2^{31}};\ \frac{1}{2_{25}};\ \frac{2^{14}}{3^{30}}\right\rangle,$$

$$F_{11} := \frac{L_{37}L_8L_4^2L_1^2}{L_{74}L_{18}L_2U(19)U(13)} = \left\langle \frac{3^9}{2^2};\ 2^{12}\cdot 3^3;\ 2^{20};\ 2^2\cdot 3^{13}\right\rangle,$$

$$F_{12} := \frac{L_{30}L_{23}L_{45}L_2^3V(19)V(13)L_1^7}{L_{92}L_{22}L_8L_3F(11)F(5)^2} = \left\langle \frac{3^{10}}{2^7};\ 2^{12}\cdot 3^5;\ 2^{27};\ 2^{18}\cdot 3^9\right\rangle.$$

Let R denote the column vector $[f(2,f(3), g(2), g(3), u(2), v(2), v(3)]$. The formulas $F_2, \ldots, F_{12}$ lead to the linear equation $MR\equiv 0 \pmod 1$, where

$$M = \begin{bmatrix} 9 & -7 & -15 & 19 & 26 & 42 & -5 \\ -1 & 2 & 9 & 6 & 25 & 11 & 13 \\ 6 & 0 & 7 & 27 & 69 & 58 & -17 \\ -2 & 1 & 8 & -4 & 1 & -9 & 6 \\ 6 & 6 & 2 & 22 & 53 & 63 & 9 \\ 4 & 0 & 5 & 5 & 19 & 7 & 8 \\ 7 & -1 & 1 & 11 & 27 & 24 & 6 \\ -5 & 7 & 11 & 3 & 22 & 14 & 8 \\ 13 & -23 & -31 & 2 & -25 & 14 & -30 \\ -2 & 9 & 12 & 3 & 20 & 2 & 13 \\ -7 & 10 & 12 & 5 & 27 & 18 & 9 \end{bmatrix}.$$

By using the Gaussian elimination over the ring of integers, we get easily that the only solution of it is $R\equiv 0 \pmod 1$. Hence, by the formulas $F(p), \ldots, V(p)$ we get immediately that $f(p), g(p), u(p), v(p)\equiv 0 \pmod 1$ for $p\leqq 19$.

References

[1] I. KÁTAI, A remark on additive functions satisfying a relation, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math., in print.*

EÖTVÖS LORÁND UNIVERSITY
DEPT. OF NUMERICAL ANALYSIS AND COMPUTER SCIENCE
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY

Acta Sci. Math., 47 (1984), 93—99

On systems of N-variable weighted shifts

B. S. YADAV and RANJANA BANSAL*)

1. Introduction. Let H be an infinite-dimensional Hilbert space with an orthonormal basis $\{e_i\}_{i=0}^{\infty}$, and let $\{w_i\}_{i=0}^{\infty}$ be a bounded sequence of complex numbers. An operator T on H defined by

$$Te_i = w_i e_{i+1}, \quad i = 0, 1, 2, \ldots$$

is called a (forward) unilateral weighted shift with the weight sequence $\{w_i\}_{i=0}^{\infty}$. JEWELL and LUBIN [2] have recently extended the theory of weighted shifts to the systems of N-variable weighted shifts. They concentrate in their paper mainly on exhibiting the interplay between such systems of operators and the analytic function theory of several variables on the lines of the survey article on (one-variable) shifts by SHIELDS [9]. The object of this note is to study the existence of cyclic vectors for the systems of N-variable backward and forward weighted shifts. We also obtain an application of these ideas to the theory of transitive operator algebras.

2. Notations and terminology. We shall follow mostly the definitions and notations given in [2]. To recapitulate, we denote by N an arbitrary but fixed positive integer, by $I=(i_1, i_2, \ldots, i_N)$ a multi-index of non-negative integers, and by $|I|$ the sum $i_1+i_2+\ldots+i_N$. E_k denotes the multi-index I having $i_k=\delta_{jk}$ for $j=1, 2, \ldots, N$, and O is the multi-index $(0, 0, \ldots, 0)$. We shall write $I\pm E_k$ for the multi-index $(i_1, \ldots, i_{k-1}, i_k\pm 1, i_{k+1}, \ldots, i_N)$ (where $i_k>0$ in $I-E_k$). If $T=\{T_1, T_2, \ldots, T_N\}$ is a system of N commuting operators on H, then, for a multi-index $I=(i_1, \ldots, i_N)$, T^I means the operator $T^I=T_1^{i_1}T_2^{i_2}\ldots T_N^{i_N}$. Let $\{e_I: I\geqq O\}$ be an orthonormal basis of H and let $\{w_{I,j}: j=1, 2, \ldots, N, I\geqq O\}$ be a bounded net of non-zero complex numbers. A system of N-variable backward weighted shifts with the weight net $\{w_{I,j}\}$ is defined as a family $S=\{S_1, \ldots, S_N\}$ of N operators on H such that $S_j e_I$ equals $w_{I,j}e_{I-E_j}$ if $i_j\neq 0$ and 0 otherwise, for all $j=1, 2, \ldots, N$.

Received October 1, 1982.

*) The second author gratefully acknowledges the support of U.G.C. grant No. F—25—9 (11633)/80 for this research work.

Similarly, a system $T=\{T_1, T_2, \ldots, T_N\}$ of N-variable forward weighted shifts with the weight net $\{w_{I,j}\}$ on H is defined by $T_j e_I = w_{I,j} e_{I+E_j}$, $1 \leqq j \leqq N$. We may and do assume $w_{I,j}$'s to be positive real numbers [2; Corollary 2]. We also assume that $w_{I,j}$'s satisfy either of the following two conditions:

(I) $$w_{I,j} w_{I-E_j,k} = w_{I,k} w_{I-E_k,j}, \quad \text{if} \quad i_j, i_k \neq 0,$$

(II) $$w_{I,j} w_{I+E_j,k} = w_{I,k} w_{I+E_k,j},$$

for all I, $1 \leqq k, j \leqq N$. As observed in [2], condition (I) (respectively condition (II)) implies that the operators S_j (respectively T_j) commute. We set $\beta(J, k, m)$ equal to $w_{J,k} w_{J-E_k,k} \ldots w_{J-(m-1)E_k,k}$ if $m \leqq j_k$, and 0 otherwise; $\alpha(J, k, m)$ equal to $w_{J,k} w_{J+E_k,k} \ldots w_{J+(m-1)E_k,k}$ if $m \neq 0$ and 1 otherwise, where $J=(j_1, j_2, \ldots, j_N)$, and $1 \leqq k \leqq N$. It is easy to see that, for $1 \leqq k \leqq N$, $S_k^m e_J$ is equal to $\beta(J, k, m) e_{J-mE_k}$ if $j_k \geqq m$ and 0 othewise, and $T_k^m e_J$ is equal to $\alpha(J, k, m) e_{J+mE_k}$.

3. Cyclic vectors for N-variable weighted shifts. We make the following definitions:

Definition 1. A net $\{w_{I,j}\}$ of (non-zero) positive real numbers is said to be monotonically decreasing if for all $I=(i_1, i_2, \ldots, i_N)$, $I'=(i_1', i_2', \ldots, i_N')$ with $|I| \geqq |I'|$, we have $w_{I,j} \leqq w_{I',j}$ for $1 \leqq j \leqq N$.

Definition 2. A vector x in H is said to be a cyclic vector for a system $A=\{A_1, A_2, \ldots, A_N\}$ of N commuting operators on H, if $H = \bigvee_I A^I x$ (i.e.: H is spanned by the vectors $A^I x$).

Theorem 1. *Let $S=\{S_1, S_2, \ldots, S_N\}$ be a system of N-variable backward weighted shifts with the weight-net $\{w_{I,j}\}$ which is monotonically decreasing and satisfies the condition* (I). *If $\{w_{I,j}\}_I$ is square-summable (i.e. $\sum_I w_{I,j}^2 < \infty$) for some j, $1 \leqq j \leqq N$, and $x = \sum_J x(J) e_J$ is any vector in H such that infinitely many elements of the set $\{x(J)\colon j_1 = j_2 = \ldots = j_N \geqq 0\}$ differ from zero, then x is a cyclic vector for S.*

Proof. For the sake of simplicity, we shall prove the theorem in case $N=2$. However, our method works also in the general case.

Assume that $\{w_{I,1}\}$ is square-summable. For given $\varepsilon > 0$, there exists an $I_0 = (i_1^0, i_2^0)$ such that

(1) $$\sum_{|J| \geqq |I_0|} w_{J,1}^2 < \varepsilon^2 (w_{(1,0),1})^2.$$

Choose $M=(m_1, m_2)$ with $m_1 \geqq i_1^0$, $m_2 \geqq i_2^0$, such that

(2) $$|x(M)| = \sup\{|x((j_1+i_1^0, j_2+i_2^0))| \colon j_1, j_2 \geqq 0\} > 0.$$

It is easy to verify, by using condition (I), that

$$w_{(1,m_2),1}\beta((0,m_2),2,m_2)=\beta((1,m_2),2,m_2)w_{(1,0),1}. \tag{3}$$

Now

$$S^M x=(S_2^{m_2}S_1^{m_1})\left(\sum x(J)e_J\right)=$$

$$=\sum x((j_1+m_1,j_2+m_2))\beta((j_1+m_1,j_2+m_2),1,m_1)\beta((j_1,j_2+m_2),2,m_2)e_J.$$

Hence, by making a successive use of (2), (3) and the fact that $\{w_{I,j}\}$ is monotonically decreasing, we have

$$\left\|\frac{S^M x}{x(M)\beta(M,1,m_1)\beta((0,m_2),2,m_2)}-e_O\right\|^2=$$

$$=\sum_{(j_1,j_2)\neq O}\left|\frac{x((j_1+m_1,j_2+m_2))}{x(M)}\right|^2\left(\frac{\beta((j_1+m_1,j_2+m_2),1,m_1)\beta((j_1,j_2+m_2),2,m_2)}{\beta((m_1,m_2),1,m_1)\beta((0,m_2),2,m_2)}\right)^2\leqq$$

$$\leqq\sum_{(j_1,j_2)\neq O}\left(\frac{\beta((j_1+m_1,j_2+m_2),1,m_1)\beta((j_1,j_2+m_2),2,m_2)}{\beta((m_1,m_2),1,m_1)\beta((0,m_2),2,m_2)}\right)^2=$$

$$=\sum_{(j_1,j_2)\neq O}\left(\frac{w_{(j_1+m_1,j_2+m_2),1}}{w_{(1,0),1}}\right)^2\times$$

$$\times\left(\frac{\beta((j_1+m_1-1,j_2+m_2),1,m_1-1)\beta((j_1,j_2+m_2),2,m_2)}{\beta((m_1,m_2),1,m_1-1)\beta((1,m_2),2,m_2)}\right)^2\leqq$$

$$\leqq\sum_{(j_1,j_2)\neq O}\left(\frac{w_{(j_1+m_1,j_2+m_2),1}}{w_{(1,0),1}}\right)^2<\varepsilon^2.$$

This shows that $e_O\in L=\bigvee_I S^I x$. Now for each $I=(i_1,i_2)$, we see that

$$y_I=S^I x-x(I)\beta(I,1,i_1)\beta((0,i_2),2,i_2)e_O$$

is in L. Using this fact and making computations such as given above, we show that $e_{(1,0)}$ is in L.

Let $N=(n_1,n_2)$ be such that

$$|x(N+E_1)|=\sup\{|x((j_1,j_2))|:\ j_1\geqq n_1+1,\ j_2\geqq n_2\}>0.$$

Proceeding as above,

$$\left\|\frac{y_N}{x(N+E_1)\beta(N+E_1,1,n_1)\beta((1,n_2),2,n_2)}-e_{(1,0)}\right\|^2=$$

$$=\sum_{(j_1,j_2)\neq O,(1,0)}\left|\frac{x((j_1+n_1,j_2+n_2))}{x((n_1+1,n_2))}\right|^2\times$$

$$\times\left(\frac{\beta((j_1+n_1,j_2+n_2),1,n_1)\beta((j_1,j_2+n_2),2,n_2)}{\beta((n_1+1,n_2),1,n_1)\beta((1,n_2),2,n_2)}\right)^2\leqq$$

$$\leqq \sum_{(j_1, j_2)\neq O, (1,0)} \left(\frac{\beta((j_1+n_1, j_2+n_2), 1, n_1)\beta((j_1, j_2+n_2), 2, n_2)}{\beta((n_1+1, n_2), 1, n_1)\beta((1, n_2), 2, n_2)} \right)^2 =$$

$$= \sum_{(j_1, j_2)\neq O, (1,0)} \left(\frac{w_{(j_1+n_1, j_2+n_2), 1}}{w_{(2,0),1}} \right)^2 \times$$

$$\times \left(\frac{\beta((j_1+n_1-1, j_2+n_2)1, n_1-1)\beta((j_1, j_2+n_2), 2, n_2)}{\beta((n_1+1, n_2), 1, n_1-1)\beta((2, n_2), 2, n_2)} \right)^2 \leqq$$

$$\leqq \frac{1}{w^2_{(2,0),1}} \sum_{(j_1, j_2)\neq O, (1,0)} w^2_{(j_1+n_1, j_2+n_2), 1} \to 0 \quad \text{as} \quad |N| \to \infty.$$

This implies that $e_{(1,0)} \in L$. Similarily, we can show that $e_{(0,1)} \in L$. Continuing this process, we see that $e_I \in L$ for all $I \geqq 0$, and hence x is a cyclic vector for S.

Theorem 2. *Let $T = \{T_1, T_2, ..., T_N\}$ be a system of N-variable forward weighted shifts with the weight-net $\{w_{I,j}\}$ which is monotonically decreasing and satisfies condition* (II). *If $\{w_{I,j}\}_I$ is square-summable for some j, $1 \leqq j \leqq N$, then any vector $x = \sum_J x(J) e_J$ in H with $x(O) \neq 0$ is a cyclic vector for T.*

Proof. Again, we prove the theorem for $N=2$. Let the net $\{w_{I,1}\}_I$ be square-summable. It can be easily seen from condition (II) that

$$\alpha(O, 2, i_2) w_{(1, i_2), 1} = w_{O,1} \alpha((1, 0), 2, i_2). \tag{4}$$

Now

$$T^I x = (T_1^{i_1} T_2^{i_2})(\sum x(J) e_J) =$$

$$= \sum x((j_1, j_2)) \alpha((j_1, j_2), 2, i_2) \alpha((j_1, j_2+i_2), 1, i_1) e_{(j_1+i_1, j_2+i_2)},$$

and hence, using successively (4) and the fact that $\{w_{I,j}\}$ is monotonically decreasing, we have

$$\left\| \frac{T^I x}{x(O)\alpha(O, 2, i_2)\alpha((0, i_2), 1, i_1)} - e_I \right\|^2 =$$

$$= \sum_{(j_1, j_2)\neq O} \left| \frac{x((j_1, j_2))}{x(O)} \right|^2 \left(\frac{\alpha((j_1, j_2), 2, i_2)\alpha((j_1, j_2+i_2), 1, i_1)}{\alpha(O, 2, i_2)\alpha((0, i_2), 1, i_1)} \right)^2 =$$

$$= \left(\frac{w_{I,1}}{w_{O,1}} \right)^2 \sum_{(j_1, j_2)\neq O} \left| \frac{x((j_1, j_2))}{x(O)} \right|^2 \left(\frac{\alpha((j_1, j_2), 2, i_2)\alpha((j_1, j_2+i_2), 1, i_1)}{\alpha((1, 0), 2, i_2)\alpha((1, i_2), 1, i_1)} \right)^2 \leqq$$

$$\leqq \left(\frac{w_{I,1}}{w_{O,1}} \right)^2 \sum_{(j_1, j_2)\neq O} \left| \frac{x((j_1, j_2))}{x(O)} \right|^2 \leqq \left(\frac{\|x\|}{|x(O)| w_{O,1}} \right)^2 w^2_{I,1}.$$

Since $\{w_{I,1}\}_I$ is square-summable and $\{e_I\}_I$ is an orthonormal basis of H, using the fact tha tthe vectors $\{T^I x\}_I$ are linearly independent, it follows by the Paley—Wiener Theorem [8] that x is a cyclic vector for T.

4. A partial solution of the transitive algebra problem. Let $\mathfrak{A}$ denote a weakly closed subalgebra of $B(H)$, the Banach algebra of all bounded operators on H. We shall write Lat $\mathfrak{A}$ for the lattice of all invariant subspaces of $\mathfrak{A}$, and shall call $\mathfrak{A}$ transitive if Lat $\mathfrak{A}=\{\{0\}, H\}$. It is easy to see that $B(H)$ is a transitive algebra. Whether there exist transitive algebras other than $B(H)$ is an open question, known as the transitive algebra problem, raised by KADISON [3] first in 1955. Since then, although the problem remains unsolved in general, a number of partial solutions of the problem have been obtained by a number of mathematicians, based mainly on the first partial solution given by ARVESON [1]. LOMONOSOV [4], however, used different techniques to obtain a partial solution of far reaching consequences. For an elegant account of these results we refer to RADJAVI and ROSENTHAL [7], see also PEARCY and SHIELDS [6]. We prove here the following theorem which generalizes a result due to NORDGREN, RADJAVI and ROSENTHAL [5]:

Theorem 3. *If a transitive algebra $\mathfrak{A}$ contains a system of N-variable backward weighted shifts S with the monotonically decreasing weight-net $\{w_{I,j}\}$ and $\{w_{I,j}\}_I$ is square-summable for some j, $1\leqq j\leqq N$, then $\mathfrak{A}=B(H)$.*

In order to prove the theorem we shall need the following lemma which is a generalization of Corollary 1 of [5; p. 176]. Let us denote by $B^{(n)}$ the direct sum of n copies of an operator B on H.

Lemma. *If $\mathfrak{A}$ is a transitive algebra containing operators A and B on H such that*
(i) *every common eigenspace of A and B is one-dimensional,*
(ii) *for each $n>0$, every non-zero common invariant subspace of $A^{(n)}$ and $B^{(n)}$ contains a common eigenvector,*
then $\mathfrak{A}$ is $B(H)$.

In the proof of the theorem we consider again only the case $N=2$. Thus we assume that $\mathfrak{A}$ contains the family $S=\{S_1, S_2\}$. Assume that $\{w_{I,1}\}_I$ is square-summable. In the light of above Lemma, it is sufficient to show that S_1 and S_2 satisfy conditions (i) and (ii).

Let L be a common eigenspace of S_1 and S_2. Then $L=\{x\in H\colon S_1x=\lambda x$ and $S_2x=\mu x\}$ for some complex numbers λ,μ. For any vector $x=\sum x(J)e_J$ in L, we have

$$\sum_{j_1\neq 0} x((j_1,j_2))w_{(j_1,j_2),1}e_{(j_1-1,j_2)} = \lambda\left(\sum x((j_1,j_2))e_{(j_1,j_2)}\right),$$

$$\sum_{j_2\neq 0} x((j_1,j_2))w_{(j_1,j_2),2}e_{(j_1,j_2-1)} = \mu\left(\sum x((j_1,j_2))e_{(j_1,j_2)}\right),$$

which implies that

$$\lambda x((j_1,j_2)) = x((j_1+1,j_2))w_{(j_1+1,j_2),1},\quad \mu x((j_1,j_2)) = x((j_1,j_2+1))w_{(j_1,j_2+1),2}$$

if $j_1, j_2 \geqq 0$. This leads to

$$x((j_1, j_2)) = \lambda^{j_1}\mu^{j_2}[\beta((j_1, j_2), 1, j_1)\beta((0, j_2), 2, j_2)]^{-1}x((0, 0))$$

for $(j_1, j_2) \neq O$. Thus $x = x((0, 0))y$, where

$$y = \sum \lambda^{j_1}\mu^{j_2}[\beta((j_1, j_2), 1, j_1)\beta((0, j_2), 2, j_2)]^{-1}e_{(j_1, j_2)}$$

is a fixed vector in H, and hence L is one-dimensional.

Next, let M be a non-zero common invariant subspace of $S_1^{(n)}$ and $S_2^{(n)}$, for $n > 0$. Let $x_1 \oplus x_2 \oplus \dots \oplus x_n$ be any non-zero element of M, where $x_k = \sum x(k, J)e_J$, $1 \leqq k \leqq n$. If for each k the net $\{x(k, J)\}_J$ has only finitely many non-zero elements, then the common invariant subspace M_1 of $S_1^{(n)}$ and $S_2^{(n)}$ generated by $x_1 \oplus x_2 \oplus \dots \oplus x_n$ is finite-dimensional. Therefore, M_1, and consequently M, has a common eigenvector of $S_1^{(n)}$ and $S_2^{(n)}$. We therefore assume that for every $I = (i_1, i_2)$, we have

$$\max\{|x(k, (j_1 + i_1, j_2 + i_2))| : j_1, j_2 \geqq 0, \ 1 \leqq k \leqq n\} > 0.$$

Let $I = (i_1, i_2)$ be fixed. Then there is a multi-index $R = R(I) = (r_1, r_2)$, $r_1 \geqq i_1$, $r_2 \geqq i_2$ and a number $s = s(I)$ between 1 and n such that

$$|x(s, R)| = \max\{|x(k, (j_1 + i_1, j_2 + i_2))| : j_1, j_2 > 0, \ 1 \leqq k \leqq n\}. \tag{5}$$

Now $(S^{(n)})^R \left(\bigoplus_{k=1}^{n} x_k\right) = \bigoplus_{k=1}^{n} S_2^{r_2} S_1^{r_1} x_k$ gives us that

$$(S^{(n)})^R(x_1 \oplus x_2 \oplus \dots \oplus x_n)[x(s, R)\beta(R, 2, r_2)\beta((r_1, 0), 1, r_1)]^{-1} =$$

$$= \left[(x(s, R))^{-1} \bigoplus_{k=1}^{n} x(k, R)e_O\right] + \left[\bigoplus_{k=1}^{n} y(k, R)\right],$$

where for each k

$$y(k, R) =$$

$$= \sum_{(j_1, j_2) \neq O} \left(\frac{x(k, (j_1 + r_1, j_2 + r_2))\beta((j_1 + r_1, j_2 + r_2), 1, r_1)}{x(s, R)\beta((r_1, r_2), 1, r_1)} \frac{\beta((j_1, j_2 + r_2), 2, r_2)}{\beta((0, r_2), 2, r_2)}\right) e_{(j_1, j_2)}.$$

Making use of (5), (I) and the facts that the net $\{w_{I,j}\}$ is monotonically decreasing and $\{w_{I,1}\}_I$ is square-summable, we have

$$\|y(k, R)\|^2 =$$

$$= \sum_{(j_1, j_2) \neq O} \left|\frac{x(k, (j_1 + r_1, j_2 + r_2))}{x(s, (r_1, r_2))}\right|^2 \left(\frac{\beta((j_1 + r_1, j_2 + r_2), 1, r_1)\beta((j_1, j_2 + r_2), 2, r_2)}{\beta((r_1, r_2), 1, r_1)\beta((0, r_2), 2, r_2)}\right)^2 \leqq$$

$$\leqq \sum_{(j_1, j_2) \neq O} \left(\frac{w_{(j_1 + r_1, j_2 + r_2), 1}}{w_{(1, 0), 1}}\right)^2 \left(\frac{\beta((j_1 + r_1 - 1, j_2 + r_2), 1, r_1 - 1)\beta((j_1, j_2 + r_2), 2, r_2)}{\beta((r_1, r_2), 1, r_1 - 1)\beta((1, r_2), 2, r_2)}\right)^2 \leqq$$

$$\leqq \frac{1}{w_{(1,0),1}^2} \sum_{(j_1, j_2) \neq O} w_{(j_1 + r_1, j_2 + r_2), 1}^2 \to 0 \quad \text{as} \quad |I| \to \infty.$$

Thus the vector $[x(s, R)]^{-1} \bigoplus_{k=1}^{n} x(k, R)e_O$ is in M. Now for each k, the net

$$\{x(k, R)/x(s, R)\colon R = R(I),\ I = (i_1, i_2)\}$$

is contained in the unit disc, and therefore contains a subnet convergent to a number z_k (say). It can be easily seen that there is a number k_0 $(1 \leqq k_0 \leqq n)$ such that k_0 will occur infinitely many times as a value of $s=s(I)$. Corresponding to this k_0, we have $z_{k_0}=1$. Thus $z_1e_O \oplus \ldots \oplus z_ne_O$ is a common eigenvector of $S_1^{(n)}$ and $S_2^{(n)}$ in M.

Lastly we make the following remarks:

1. The condition of monotonicity of the net $\{w_{I,j}\}$ in all our theorems can be replaced by the following less stringent requirements: if j_0 is the integer for which $\{w_{I,j}\}_I$ is square-summable, then

(i) $w_{(i_1, i_2, \ldots, i_N), j} \leqq w_{(i_1', i_2', \ldots, i_N'), j}$ for $1 \leqq j \leqq N$ if $i_k \geqq i_k'$, $1 \leqq k \leqq N$,

(ii) $w_{(i_1, \ldots, i_{j_0-1}, i_{j_0}-1, i_{j_0+1}, \ldots, i_N), j} \leqq w_{(i_1', \ldots, i_{j_0-1}', i_{j_0}', i_{j_0+1}', \ldots, i_N'), j}$ for $1 \leqq j \leqq N$ if $i_k \geqq i_k'$, $1 \leqq k \leqq N$, $k \neq j_0$, $i_{j_0} \geqq 1$ ($i_{j_0}=1$ in Theorem 2, Theorem 3).

2. We observe that Theorem 2 and Theorem 3 can be proved in a more general form on the lines of Theorem 1 in [10] and Theorem 1 in [11] respectively.

References

[1] W. B. ARVESON, A density theorem for operator algebras, *Duke Math. J.*, **34** (1967), 635—647.

[2] N. P. JEWELL and A. R. LUBIN, Commuting weighted shifts and analytic function theory in several variables, *J. Operator Theory,* **1** (1979), 207—233.

[3] R. V. KADISON, On orthogonalization of operator representations, *Amer. J. Math.*, **78** (1955), 600—621.

[4] V. J. LOMONOSOV, Invariant subspaces for operators commuting with compact operators, *Functional Anal., Appl.* 7 (1973), 213—214.

[5] E. NORDGREN, H. RADJAVI and P. ROSENTHAL, On density of transitive algebras, *Acta Sci. Math.*, **30** (1969), 175—179.

[6] C. PEARCY and A. L. SHIELDS, A Survey of Lomonosov technique in the theory of invariant subspaces, in: *Topics in Operator Theory,* Amer. Math. Soc. Surveys, No. 13 (Providence, R. I., 1974).

[7] H. RADJAVI and P. ROSENTHAL, *Invariant Subspaces,* Springer-Verlag (New York, 1973).

[8] F. RIESZ and B. SZ.-NAGY, *Functional Analysis,* Ungar (New York, 1955).

[9] A. L. SHIELDS, Weighted shift operators and analytic function theory, in: *Topics in Operator Theory,* Amer. Math. Soc. Surveys, No. 13 (Providence, R. I., 1974).

[10] B. S. YADAV and S. CHATTERJEE, On a characterization of invariant subspace lattices of weighted shifts, *Proc. Amer. Math. Soc.*, **84** (1982), 442—446.

[11] B. S. YADAV and S. CHATTERJEE, A partial solution of transitive algebra problem, *Acta Sci. Math.*, **42** (1980), 211—215.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DELHI
DELHI—110007, INDIA

Acta Sci. Math., 47 (1984), 101—106

Moment theorems for operators on Hilbert space. II

ZOLTÁN SEBESTYÉN

Introduction. The present note is a direct continuation of our previous investigation [1] about the momentlike problems of the existence of a contraction or a subnormal operator T on Hilbert space H such that $x_n=T^n x_0$ $(n=1, 2, \ldots)$ for some given sequence $\{x_n\}_{n=0}^{\infty}$ in H. The corresponding continuous problem was to find a continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions on H such that $T_0=I_H$ and $x_t=T_t x_0$ $(t\geqq 0)$ with some given continuous family $\{x_t\}_{t\geqq 0}$ in H. Our present object is to generalize these problems as follows.

Problems. Given a sequence $\{A_n\}_{n=0}^{\infty}$ of bounded linear operators on H it is natural to ask: under what condition does there exist an operator T on H with

$$(1) \qquad A_n = T^n A_0 \quad (n = 1, 2, \ldots).$$

For a continuous family $\{A_t\}_{t\geqq 0}$ of operators on H a continuous semigroup $\{T_t\}_{t\geqq 0}$ of bounded linear operators with $T_0=I_H$ and

$$(2) \qquad A_t = T_t A_0 \quad (t \geqq 0)$$

may be sought.

We shall treat only the following cases:

(A) (1) holds with a contraction T.

(B) (2) holds with a continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions such that $T_0=I_H$.

(C) (1) holds with a subnormal operator T.

Results.

Theorem A. *Problem* (A) *has a solution if and only if*

Received March 5, 1982, and in revised form February 22, 1983.

$$\|\sum_{n',n} A_{n'+n}h_{n',n}\|^2 \leqq \sum_{\substack{m\geqq n\\ m',n'}} (A_{m'-n+m}h_{m',m}, A_{n'}h_{n',n}) + \tag{3}$$

$$+\sum_{\substack{n<m\\ m',n'}} (A_{m'}h_{m',m}, A_{n'-m+n}h_{n',n})$$

holds for any finite sequence $\{h_{n',n}\}_{n'\geqq 0, n\geqq 0}$ *in* H.

Theorem B. *Problem* (B) *has a solution if and only if*

$$\|\sum_{t',t} A_{t'+t}h_{t',t}\|^2 \leqq \sum_{\substack{t\leqq s\\ s',t'}} (A_{s'-t+s}h_{s's}, A_{t'}h_{t',t}) + \sum_{\substack{s<t\\ s',t'}} (A_{s'}h_{s',s}, A_{t'-s+t}h_{t',t}) \tag{4}$$

holds for any finite sequence $\{h_{t',t}\}_{t'\geqq 0, t\geqq 0}$ *in* H.

Theorem C. *Let* $\{A_n\}_{n=0}^{\infty}$ *be a sequence of operators on the Hilbert space* H *such that*

(i) $\{\text{Range } A_n\}_{n=0}^{\infty}$ *spans the space* H.

(ii) $\|A_n\| \leqq \varkappa^n$ *holds for some constant* $\varkappa \geqq 0$ *and* $n=1, 2, \dots$.

Under these assumptions Problem (C) *has a solution if and only if there exists a double sequence* $\{A_{n',n}\}_{n'=0, n=0}^{\infty}$ *of operators on* H *such that*

$$A_{0,n} = A_n \quad \textit{for} \quad n = 0, 1, 2, \dots, \tag{iii}$$

$$A_m^* A_{n',n} = A_{n'+m}^* A_n \quad \textit{for} \quad m, n', n = 0, 1, 2, \dots, \quad \textit{and} \tag{iv}$$

$$\|\sum_{n',n} A_{n',n}h_{n',n}\|^2 \leqq \sum_{\substack{n',m\\ n',n}} (A_{n'+m}h_{m',m}, A_{m'+n}h_{n',n}) \tag{v}$$

hold for any finite (double) sequence $\{h_{n',n}\}_{n'=0, n=0}$ *in* H.

Necessity. (A) Let U be a unitary dilation (see [2]) of T on some Hilbert space K containing H. Then

$$PU^n h = T^n h \quad (h\in H;\ n = 0, 1, 2, \dots), \tag{5}$$

where P is the orthogonal projection of K onto H. Let further $\{h_{n',n}\}_{n'=0, n=0}$ be any finite (double) sequence in H. Then by (1) and (5)

$$\|\sum_{n',n} A_{n'+n}h_{n',n}\|^2 = \|\sum_{n',n} T^n A_{n'}h_{n',n}\|^2 \leqq \|\sum_{n',n} U^n A_{n'}h_{n',n}\|^2 =$$

$$= \sum_{\substack{n\leqq m\\ m',n'}} (U^{m-n}A_{m'}h_{m',m}, A_{n'}h_{n',n}) + \sum_{\substack{m<n\\ m',n'}} (A_{m'}h_{m',m}, U^{n-m}A_{n'}h_{n',n}) =$$

$$= \sum_{\substack{n\leqq m\\ m',n'}} (T^{m-n}A_{m'}h_{m',m}, A_{n'}h_{n',n}) + \sum_{\substack{m<n\\ m',n'}} (A_{m'}h_{m',m}, T^{n-m}A_{n'}h_{n',n}) =$$

$$= \sum_{\substack{n\leqq m\\ m',n'}} (A_{m'-n+m}h_{m',m}, A_{n'}h_{n',n}) + \sum_{\substack{m<n\\ m',n'}} (A_{m'}h_{m',m}, A_{n'-m+n}h_{n',n}).$$

(B) Let U_t be a unitary dilation (see [2]) of the continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions on some Hilbert space K containing H. Then

$$PU_t h = T_t h \quad (h\in H;\ t\geqq 0),$$

where P is the orthogonal projection of K onto H. For any finite (double) sequence $\{h_{t',t}\}_{t'\geqq 0, t\geqq 0}$ in H (4) can be verified in the same manner as (3) was before.

(C) Let N be a normal extension of T on a Hilbert space K containing H. Then

$$PN^{*n'}N^n h = T^{*n'}T^n h \quad (h\in H;\ n', n = 0, 1, 2, \dots), \tag{6}$$

where P denotes the orthogonal projection of K onto H. Let further

$$A_{n',n} = T^{*n'}T^n A_0 \quad (n', n = 0, 1, 2, \dots).$$

Then by (1) and (6) we have $A_{0,n}=T^{*0}T^nA_0=T^nA_0=A_n$ for $n=0,1,2,\dots$. Furthermore, for any h, k in H

$$(A_m^* A_{n',n}h, k) = (T^{*n'}T^nA_0h, A_mk) =$$
$$= (T^{*(n'+m)}T^nA_0h, A_0k) = (N^{*(n'+m)}N^nA_0h, A_0k) =$$
$$= (N^nA_0h, N^{n'+m}A_0k) = (T^nA_0h, T^{n'+m}A_0k) =$$
$$= (A_nh, A_{n'+m}k) = (A_{n'+m}^*A_nh, k).$$

Finally, for any finite (double) sequence $\{h_{n',n}\}_{n'=0,n=0}$ in H we have

$$\|\sum_{n',n} A_{n',n}h_{n',n}\|^2 = \|P\sum_{n',n} N^{*n'}N^nA_0h_{n',n}\|^2 \leqq \|\sum_{n',n} N^{*n'}N^nA_0h_{n',n}\|^2 =$$
$$= \sum_{\substack{m',m\\ n',n}} (N^{n'+m}A_0h_{m',m}, N^{m'+n}A_0h_{n',n}) = \sum_{\substack{m',m\\ n',n}} (T^{n'+m}A_0h_{m',m}, T^{m'+n}A_0h_{n',n}) =$$
$$= \sum_{\substack{m',m\\ n',n}} (A_{n'+m}h_{m',m}, A_{m'+n}h_{n',n}).$$

These prove the properties (iii), (iv), (v) as required.

Sufficiency. (A) Let F_0 be the linear space of all double sequences $\{h_{n',n}\}$ $(n', n= =0, 1, 2, \dots)$ in the Hilbert space H. In view of (3) one can define a semi-definite inner product $\langle\cdot,\cdot\rangle$ (for $\{h_{m',m}\}$, $\{k_{n',n}\}$ in F_0) by

$$\langle\{h_{m',m}\}, \{k_{n',n}\}\rangle := \sum_{\substack{n\leqq m\\ m',n'}} (A_{m'-n+m}h_{m',m}, A_{n'}k_{n',n}) + \sum_{\substack{m<n\\ m',n'}} (A_{m'}h_{m',m}, A_{n'-m+n}k_{n',n}).$$

Hence we obtain a Hilbert space F by factoring F_0 with respect to the null space of $\langle\cdot,\cdot\rangle$ and completing this factor space with respect to the norm arising from the new definite inner product (denoted also by $\langle\cdot,\cdot\rangle$). For simplicity the residue class of $\{h_{m',m}\}\in F_0$ in F is denoted by the same symbol. We have two natural operations on

F_0: for $\{h_{n',n}\}$ in F_0 we set

$$U_0\{h_{n',n}\} = \{h'_{n',n}\} \quad \text{where} \quad h'_{n',n} = h_{n',n-1} \quad \text{for} \quad n \geqq 1 \quad \text{and} \quad h'_{n',0} = 0;$$

$$V_0\{h_{n',n}\} = \sum_{n',n} A_{n'+n} h_{n',n}.$$

U_0 and V_0 induce an isometry U of F into F and a contraction V of F into H, as an easy calculation shows. We are going to show that $T=VUV^*$ is the desired contraction on H.

Indeed, for any $\{h_{m',m}\}$ in F_0 and h in H

$$\langle\{h_{m',m}\}, V^*A_jh\rangle = (V\{h_{m',m}\}, A_jh) = \sum_{m',m} (A_{m'+m}h_{m',m}, A_jh) =$$

$$= \langle\{h_{m',m}\}, \{k^{(j)}_{n',n}\}\rangle \quad (j = 0, 1, 2, \ldots),$$

where $k^{(j)}_{n',n}=h$ if $n=0$, $n'=j$, and 0 otherwise. Hence $V^*A_jh=\{k^{(j)}_{n',n}\}$, and thus $UV^*A_jh=\{k^{(j)'}_{n',n}\}$, where $k^{(j)'}_{n',n}=h$ if $n=1, n'=j$, and 0 otherwise. It follows that for h in H, $j=0, 1, 2, \ldots$,

$$TA_jh = VUV^*A_jh = \sum_{n',n} A_{n'+n}k^{(j)'}_{n',n} = A_{j+1}h.$$

This is actually identical with (1), and the proof is complete.

(B) The proof is completely analogous to the previous one. The continuity of the family of contractions is a direct consequence of the continuity of the original operator family and the construction given in the proof.

(C) Let $\{A_{n',n}\}$ $(n', n=0, 1, \ldots)$ be a double sequence of operators satisfying (i)—(v) of Theorem C on the Hilbert space H. Let K_0 be the linear space of all finite double sequences $\{h_{n',n}\}$ $(n', n=0, 1, \ldots)$ in H with semi-definite inner product of $\{h_{m',m}\}$ and $\{k_{n',n}\}$ in K_0 given by

$$\langle\{h_{m',m}\}, \{k_{n',n}\}\rangle := \sum_{\substack{m',m\\ n',n}} (A_{n'+m}h_{m',m}, A_{m'+n}k_{n',n}). \tag{7}$$

As in the proof of Theorem A we get a Hilbert space K in which the elements of K_0 may be considered to form a dense linear manifold. We also have two operations on K_0, namely for $\{h_{n',n}\}$ in K_0 we set

$$N_0\{h_{n',n}\} = \{h^{0;1}_{n',n}\} \quad \text{where} \quad h^{0;1}_{n',n} = h_{n',n-1} \quad \text{for} \quad n \geqq 1 \quad \text{and} \quad h^{0;1}_{n',0} = 0.$$

$$V_0\{h_{n',n}\} = \sum_{n',n} A_{n',n}h_{n',n}.$$

In view of (v), V_0 induces a contraction V of K into H.

By (7) for any $\{h_{m',m}\}$ in K_0 and h in H

$$\langle\{h_{m',m}\}, V^*A_jh\rangle = (V\{h_{m',m}\}, A_jh) =$$

$$= \sum_{m',m} (A_{m',m}h_{m',m}, A_jh) = \langle\{h_{m',m}\}, \{k^{(j)}_{n',n}\}\rangle \quad (j = 0, 1, 2, \ldots),$$

that is,

(8) $V^* A_j h = \{k_{n',n}(j)\}$ where $k_{n',n}(j)=h$ if $n=j$, $n'=0$, and 0 otherwise.

It follows that $VV^* A_j h = A_{0,j} h = A_j h$, and hence by (i)

(9) $$VV^* = I_H \text{ (=the identity on } H).$$

That N_0 induces a normal operator on H needs some further argument.

First of all we show that $N_0^* \{k_{n',n}\} = \{k_{n',n}^{1,0}\}$ where $k_{n',n}^{1,0} = k_{n'-1,n}$ for $n' \geqq 1$ and $k_{0,n}^{1,0}=0$. This follows from (7) since for any $\{h_{m',m}\}$ in K_0 we have

$$\langle \{h_{m',m}\}, N_0^* \{k_{n',n}\}\rangle = \langle N_0 \{h_{m',m}\}, \{k_{n',n}\}\rangle =$$
$$= \sum_{\substack{m',m\\ n',n}} (A_{n'+m+1} h_{m',m}, A_{m'+n} k_{n',n}) = \langle \{h_{m',m}\}, \{k_{n',n}^{1,0}\}\rangle.$$

Let $h_{n',n}^{j',j} = h_{n'-j',n-j}$ for $n' \geqq j'$, $n \geqq j$, and 0 otherwise. Then

$$(N_0^* N_0)^{2^j} \{h_{n',n}\} = \{h_{n',n}^{2^j,2^j}\} \quad \text{for any} \quad \{h_{n',n}\} \quad \text{in} \quad K_0.$$

Hence

$$\|N_0 \{h_{n',n}\}\|^2 = \langle N_0^* N_0 \{h_{n',n}\}, \{h_{n',n}\}\rangle \leqq \|N_0^* N_0 \{h_{n',n}\}\| \cdot \|\{h_{n',n}\}\| \leqq$$
$$\leqq \|(N_0^* N_0)^2 \{h_{n',n}\}\|^{1/2} \cdot \|\{h_{n',n}\}\|^{3/2}$$

and by induction, for any $j=0, 1, 2, \ldots,$

$$\|N_0 \{h_{n',n}\}\|^{2^{j+2}} \leqq \|(N_0^* N_0)^{2^j} \{h_{n',n}\}\|^2 \cdot \|\{h_{n',n}\}\|^{2^{j+2}-2} =$$
$$= \|\{h_{n',n}^{2^j,2^j}\}\|^2 \cdot \|\{h_{n',n}\}\|^{2^{j+2}-2} =$$
$$= \|\{h_{n',n}\}\|^{2^{j+2}-2} \sum_{\substack{m',m\\ n',n}} (A_{n'+m+2^j} h_{m',m}, A_{m'+n+2^j} h_{n',n}) \leqq$$
$$\leqq \|\{h_{n',n}\}\|^{2^{j+2}-2} \sum_{\substack{m',m\\ n',n}} \|A_{n'+m+2^j}\| \cdot \|A_{m'+n+2^j}\| \cdot \|h_{m',m}\| \cdot \|h_{n',n}\| \leqq$$
$$\leqq \|\{h_{n',n}\}\|^{2^{j+2}-2} \varkappa^{2^{j+1}} \Big(\sum_{m',m} \varkappa^{m'+m} \|h_{m',m}\|\Big)^2.$$

Letting $j \to \infty$ we get $\|N_0 \{h_{n',n}\}\| \leqq \sqrt{\varkappa}\, \|\{h_{n',n}\}\|$ for any $\{h_{n',n}\}$ in K_0. Thus N_0 can be extended by continuity to a normal operator N on K.

We shall show that $T = VNV^*$ is the desired subnormal operator on H. In view of (9), we may regard N as a normal extension of T, only H must be identified (by the aid of V^*) with a subspace of K. Finally (8) implies

$$TA_j h = VNV^* A_j h = VN\{k_{n',n}(j)\} = V\{k_{n',n}^{0,1}(j)\} = A_{j+1} h$$

for any h in H and $j=0, 1, 2, \ldots,$ which amounts to (1).

The proof is complete.

We remark that

$$T^*A_jh = VN^*V^*A_jh = VN^*\{k_{n',n}(j)\} = V\{k^{1;0}_{n',n}\} = A_{1,j}h$$

for any h in H and $j=0, 1, 2, \ldots$.

The method of proof of Theorem C yields also the following.

Proposition. *Let* $\{A_{n',n}\}$ $(n', n=0, 1, 2, \ldots)$ *be a double sequence of operators on the Hilbert space H such that* $\{\text{Range } A_{0,n}\}_{n=0}^{\infty}$ *spans H. There exists a normal operator T on H with*

(1*) $$A_{n',n} = T^{*n'}T^nA_0 \quad \textit{for} \quad n', n = 0, 1, 2, \ldots$$

if and only if

(iv**) $A^*_{n',n}A_{m',m} = A^*_{0,m'+n}A_{0,n'+m}$ *for* $m', m, n', n=0, 1, \ldots,$ *and*

(ii*) *there exists a constant* $\varkappa' \geqq 0$ *such that* $\|A_{n',n}\| \leqq \varkappa'^{n'+n}$ $(n', n=0, 1, 2, \ldots)$.

Proof. The necessity of the condition is simple so that we omit the details. Assume, on the contrary, that (ii*) and (iv*) are satisfied. Set $A_n = A_{0,n}$ for $n=0, 1, 2, \ldots$. Then we have equality in (v) of Theorem C, and therefore V is an isometry of K into H (see the proof of Theorem C) so that by (9) V is unitary between K and H. As a consequence, $T=VNV^*$ is unitarily equivalent to the normal operator N, hence is itself normal and by (10), satisfies (1*). The proof is complete.

References

[1] Z. Sebestyén, Moment theorems for operators on Hilbert space, *Acta Sci. Math.*, **44** (1982), 165—171.

[2] B. Sz.-Nagy, Extensions of linear transformations in Hilbert space with extend beyond this space, Appendix to F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar (New York, 1960).

DEPARTMENT OF MATH. ANALYSIS II
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KÖRÚT 6—8
1088 BUDAPEST, HUNGARY

Acta Sci. Math., 47 (1984), 107—111

J-unitary equivalence of positive subspaces of a Krein space

N. J. YOUNG

A J-unitary operator on a Krein space preserves the indefinite metric and hence maps positive subspaces to positive subspaces. The purpose of this note is to answer the following question: if L, M are maximal positive subspaces of a Krein space, under what conditions does there exist a J-unitary operator U such that $UL=M$? The answer turns out to be pleasantly concise. Note an obvious necessary condition: L and M must be isometrically isomorphic with respect to the semi-norms induced on them by the indefinite metric, and the same goes for their orthogonal companions. For most subspaces these conditions are also sufficient for the existence of the desired U, but there is an exceptional class for which an extra condition, related to the Fredholm index, is needed.

It is well known [3, 4] that questions on the action of J-unitaries on maximal positive subspaces can be re-formulated in terms of symplectic transformations of contraction operators, and in fact the theorems below are little more than a re-interpretation in the context of Krein spaces of recent results of the author's on orbits of the symplectic group [5].

Let H be a Krein space with fundamental symmetry J. That is, H is a Hilbert space with inner product $(\cdot, \cdot)$, having a distinguished orthogonal decomposition $H=H_+ \oplus H_-$, and $J=P_+ - P_-$ where P_+, P_- are the orthogonal projections on H_+, H_- respectively. We introduce an indefinite inner product $[\cdot, \cdot]$ on H by $[x, y]= =(Jx, y)$, and we call an operator U on H *J-unitary* if U is surjective and $[Ux, Uy]= =[x, y]$ for all $x, y \in H$, or equivalently, if

$$U^* JU = J = UJU^*. \tag{1}$$

A subspace L of H is said to be *positive* if $[x, x] \geqq 0$ for all $x \in L$, and to be *maximal positive* if it is positive and is not a proper subset of any positive subspace. Likewise one defines *negative* and *maximal negative* subspaces. If L is any subspace of H then

Received June 2, 1982.

the *orthogonal companion* L^J of L is the subspace

$$L^J = \{x \in H: [x, y]=0 \quad \text{for all} \quad y \in L\}.$$

The orthogonal companion of a maximal positive subspace is a maximal negative subspace. If L is a positive or negative subspace then the indefinite metric of H induces a semi-norm p on L by $p(x)=|[x, x]|^{1/2}$; this will be called the *intrinsic semi-norm* of L.

If U is J-unitary and $UL=M$ for some positive subspace L, then clearly M is positive and L, M are isometrically isomorphic with respect to their intrinsic semi-norms. Moreover, $U(L^J)=M^J$ and L^J, M^J are also isometrically isomorphic. As was indicated above, this pair of isometric isomorphisms does not in general suffice for the existence of a J-unitary U such that $UL=M$, and we are obliged to introduce another condition.

We shall say that a maximal positive subspace L of H is *Fredholm* if P_-L is closed in H_- and both $L \cap H_+$ and $L^J \cap H_-$ are finite-dimensional. We define the *signature* of a Fredholm positive subspace L to be $\dim (L \cap H_+) - \dim (L^J \cap H_-)$.

Note that if H is finite-dimensional then every positive subspace of H is Fredholm and has signature $\dim H_+ - \dim H_-$, which is the signature of the Hermitian form $[x, x]$ in the classical sense of Sylvester.

We shall say that a topological vector space E is *permanently incomplete* if it contains no complete infinite-dimensional subspace modulo the closure of $\{0\}$: that is, if E/E_0 with the quotient topology contains no complete infinite-dimensional subspace, where E_0 is the closure of $\{0\}$ in E.

Theorem. *Let H be a separable Krein space with fundamental symmetry J and let L, M be maximal positive subspaces of H. There exists a J-unitary operator U on H such that $UL=M$ if and only if the following two conditions hold:*

(i) *L, L^J are isometrically isomorphic to M, M^J respectively in their intrinsic semi-norms, and*

(ii) *if L and L^J are permanently incomplete with respect to their intrinsic semi-norms then L and M have the same signature.*

It will transpire during the proof that if L and L^J are permanently incomplete then L is Fredholm, so that condition (ii) makes sense.

We prove the theorem using the correspondence between positive subspaces and contractions. Every maximal positive subspace L of H is the graph of a contraction from H_+ to H_-; that is,

$$L = \{\langle x_+, Kx_+ \rangle: \ x_+ \in H_+\} \tag{2}$$

where K: $H_+ \to H_-$ is a linear operator and $\|K\| \leqq 1$. K is uniquely determined by L and is called the *angle operator* of L. It is clear that, for any contraction K from H_+ to H_-, the space L defined by (2) is a maximal positive subspace of H, so there is a one-one correspondence between maximal positive subspaces of H and contractions from $H_+ \to H_-$. See [1, 3, or 4] for fuller details. If L has angle operator K then the maximal negative subspace L^J is the graph of the contraction $K^*: H_- \to H_+$. Furthermore we have

$$P_- L = \text{Range}\, K; \ \text{Ker}\, K = L \cap H_+; \ \text{Ker}\, K^* = L^J \cap H_-$$

and so L is a Fredholm subspace in the sense defined above if and only if its angle operator K is a Fredholm operator (see, for example [2]), and the signature of a Fredholm L is the Fredholm index of its angle operator.

Applying a J-unitary transformation U to a maximal positive subspace corresponds to taking a symplectic transformation of its angle operator. With respect to the orthogonal decomposition $H = H_+ \oplus H_-$ of H, U can be written as a 2×2 matrix of operators

$$U = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \tag{3}$$

where $D: H_+ \to H_+$, $C: H_- \to H_+$ etc. are linear operators. It is easily calculated [4] that if L is maximal positive with angle operator K then UL has angle operator

$$\Phi(K) = (AK+B)(CK+D)^{-1}. \tag{4}$$

A transformation of the form (4), where U given by (3) is J-unitary on H, is called a *symplectic transformation* on the space $L(H_+, H_-)$ of all bounded linear operators from H_+ to H_-. The condition (1) (that U be J-unitary) does ensure that $\Phi(K)$ be well defined — that is, $CK+D$ is invertible on H_+ for every contraction $K \in L(H_+, H_-)$ (see [3]).

In view of these correspondences we perceive that the spaces of the form UL, where U is J-unitary, are those maximal positive subspaces of H whose angle operators lie in the same orbit of the closed unit ball of $L(H_+, H_-)$ under the symplectic group as the angle operator K of L. We can therefore make use of Theorem 1 of [5] which states the following, for separable Hilbert spaces H_+, H_-.

Let $X, Y \in L(H_+, H_-)$ be contractions. There exists a symplectic transformation Φ on $L(H_+, H_-)$ such that $\Phi(X) = Y$ if and only if

(S1) *$I - Y^*Y$, $I - YY^*$ are congruent to $I - X^*X$, $I - XX^*$ respectively, and*

(S2) *if X is essentially unitary then* $\text{ind}\, X = \text{ind}\, Y$.

Here two Hermitian operators M, N on a Hilbert space G are said to be congruent if there exists an invertible operator T on G, with bounded inverse, such that $M = T^*NT$. And an operator X is said to be *essentially unitary* if $I - X^*X$ and $I - XX^*$ are compact operators.

Suppose, then, that L, M are maximal positive subspaces of H with angle operators X, Y respectively. Our discussion shows that there exists a J-unitary operator

U on H such that $UL=M$ if and only if X and Y satisfy the conditions (S1) and (S2) above. It remains to translate these into statements about the geometry of L and M.

Lemma 1. *L and M are isometrically isomorphic in their intrinsic semi-norms if and only if $I-Y^*Y$ is congruent to $I-X^*X$.*

Proof. ($\Rightarrow$) Let $R: L\to M$ be an isometric isomorphism. R maps a typical element $\langle x_+, Xx_+\rangle$ of L to an element $\langle y_+, Yy_+\rangle$ of M: write $y_+=Tx_+$. T is clearly a bijective bounded linear operator on H_+, and so it is invertible, by the closed graph theorem. The intrinsic semi-norm p on L is given by

$$p(\langle x_+, Xx_+\rangle)^2 = (x_+, x_+)-(Xx_+, Xx_+) = ((I-X^*X)x_+, x_+),$$

while for the intrinsic semi-norm q on M we have

$$q(R\langle x_+, Xx_+\rangle)^2 = q(\langle Tx_+, YTx_+\rangle)^2 = ((I-Y^*Y)Tx_+, Tx_+).$$

Hence the supposition that R be an isometry entails $I-X^*X=T^*(I-Y^*Y)T$. Conversely, if this congruence relation holds for some invertible T then the formula $R\langle x_+, Xx_+\rangle=\langle Tx_+, YTx_+\rangle$ defines an isometric isomorphism $R: L\to M$.

Lemma 2. *Let E be a Hilbert space and let T be a bounded linear operator on E. Let $p_T(x)=\|Tx\|$ for $x\in E$. Then E is permanently incomplete with respect to the semi-norm p_T if and only if T is compact.*

Proof. Let $F=E/\mathrm{Ker}\, T$ and let $K: E\to F$ be the quotient mapping. p_T induces a norm on F, which we again denote by p_T. T induces an operator T_1: $F\to E$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "K"] \arrow[dr, "T"'] & & (F, p_T) \arrow[dl, "T_1"] \\ & E & \end{tikzcd}$$

commutes. T_1 is an isometry of F with Range T. It is easy to see (using the polar decomposition and the spectral theorem) that T is compact if and only if Range T contains no closed infinite-dimensional subspace, hence if and only if (F, p_T) contains no complete infinite-dimensional subspace. This is the definition of the permanent incompleteness of (E, p_T).

Corollary. *X is essentially unitary if and only if both L and L^J are permanently incomplete in their intrinsic semi-norms.*

Proof. The proof of Lemma 1 shows that the intrinsic semi-norm p of L is given by

$$p(\langle x_+, Xx_+\rangle) = \|(I-X^*X)^{1/2}x_+\|.$$

Lemma 2 now shows that (L, p) is permanently incomplete if and only if $(I-X^*X)^{1/2}$ is compact, and this is so if and only if $I-X^*X$ is compact. Likewise L^J is permanently incomplete if and only if $I-XX^*$ is compact.

We can now conclude the proof of the Theorem. Lemma 1 shows that the angle operators X, Y of L, M satisfy condition (S1) if and only if L, L^J are isometrically isomorphic to M, M^J in their intrinsic semi-norms, which is condition (i) of the theorem. For condition (ii), the Corollary shows that L and L^J are permanently incomplete if and only if X is essentially unitary, and in this case the requirement $\operatorname{ind} X = \operatorname{ind} Y$ of (S2) is manifestly the same as the equality of the signatures of L and M.

It might be asked whether condition (ii) of the Theorem is really needed: it is conceivable that the isometric isomorphism condition (i) might imply the equality of the signatures of L and M in the permanently incomplete case. In fact it does not, as an example in [5, Section 5] shows. There exist contractions X, Y on a separable Hilbert space G satisfying the congruence conditions (S1) above, with X, Y being compact perturbations of the identity and a unilateral shift respectively. They are thus essentially unitary but of different index. If we define a Krein space H with $H_+ = G = H_-$ and take L, M to be the maximal positive subspaces with angle operators X, Y respectively then L, M satisfy condition (i) but not condition (ii) of the Theorem.

We note that Theorem 2 of [5] gives a recipe for constructing all symplectic transformations Φ such that $\Phi(X) = Y$ in terms of X, Y and the operators implementing the congruences of condition (i) of the Theorem.

References

[1] J. Bognár, *Indefinite inner product spaces,* Springer Verlag (Berlin, 1974).

[2] R. G. Douglas, *Banach algebra techniques in operator theory,* Academic Press (New York, 1972).

[3] J. W. Helton, Unitary operators on a space with indefinite inner product, *J. Funct. Anal.,* **6** (1970), 412—440.

[4] M. G. Krein, A new application of the fixed-point principle in the theory of operators on a space with indefinite metric, *Soviet Math. Dokl.,* **5** (1964), 224—228.

[5] N. J. Young, Orbits of the unit sphere of $L(H, K)$ under symplectic rransformations, *J. Operator Theory,* **11** (1984), 171—191.

DEPARTMENT OF MATHEMATICS
UNIVERSITY GARDENS
GLASGOW, G12 8QW, SCOTLAND

Acta Sci. Math., 47 (1984), 113—115

On minimal invariant manifolds and density of operator algebras

HEYDAR RADJAVI

Let $\mathfrak{X}$ be a complex Banach space and let $\mathfrak{A}$ be a subalgebra of $\mathscr{B}(\mathfrak{X})$, the algebra of all bounded linear operators on $\mathfrak{X}$. When is $\mathfrak{A}$ strongly dense in $\mathscr{B}(\mathfrak{X})$, i.e., dense in the topology of pointwise convergence? This question can sometimes be answered by examining various lattices associated with $\mathfrak{A}$. The first result of this sort was obtained by RICKART and YOOD [2, p. 62], a consequence of which is: if the only linear manifolds (i.e., not necessarily closed subspaces) of $\mathfrak{X}$ invariant under all members of $\mathfrak{A}$ are $\{0\}$ and $\mathfrak{X}$, then $\mathfrak{A}$ is strongly dense in $\mathscr{B}(\mathfrak{X})$. If we denote by $\mathscr{L}(\mathfrak{A})$ the lattice of all linear manifolds invariant under $\mathfrak{A}$, the hypothesis in this assertion amounts to saying that $\mathscr{L}(\mathfrak{A})$ is trivial, that is, the only nonzero element of $\mathscr{L}(\mathfrak{A})$ is $\mathfrak{X}$. We shall prove the following result.

Theorem 1. *Let $\mathfrak{A}$ be an algebra of bounded linear operators on the Banach space $\mathfrak{X}$. If the nonzero elements of $\mathscr{L}(\mathfrak{A})$ have a dense intersection, then $\mathfrak{A}$ is strongly dense in $\mathscr{B}(\mathfrak{X})$.*

Note that the hypothesis of the theorem implies that $\mathscr{L}(\mathfrak{A})$ has no *closed* members other than $\{0\}$ and $\mathfrak{X}$, i.e., $\mathfrak{A}$ is topologically transitive. It is not known whether topological transitivity for $\mathfrak{A}$ is sufficient for strong density if $\mathfrak{X}$ is a reflexive Banach space. (This is the Transitive Algebra Problem; see [3]).

Many examples of algebras satisfying the hypothesis of Theorem 1 exist. Here is a simple, nontrivial example. Fix an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for a separable, infinite-dimensional Hilbert space $\mathfrak{H}$. Let $\mathfrak{A}$ be the set of all those operators on $\mathfrak{H}$ whose matrices relative to $\{e_i\}_{i=1}^{\infty}$ are "column-finite", i.e., each of their columns have finitely many nonzero entries. It is easy to see that $\mathfrak{A}$ is in fact an algebra, that every operator in $\mathfrak{A}$ leaves the linear span $\mathscr{V}$ of $\{e_i\}_{i=1}^{\infty}$ invariant, and that, furthermore, $\mathscr{V}$ is contained in every invariant linear manifold of $\mathfrak{A}$. Another example is the subalgebra $\mathfrak{A}_0$ of the above $\mathfrak{A}$ consisting of finite-rank operators; $\mathscr{V}$ is still the intersection of all nonzero members of $\mathscr{L}(\mathfrak{A}_0)$.

Received July 19, 1982.

To prove Theorem 1, we shall need a lemma, which seems to be of independent interest, and which is itself an extension of the Rickart—Yood result referred to above. We first recall some definitions: An algebra $\mathfrak{A}$ of linear operators on any vectoı space $\mathscr{V}$ is called *strictly transitive* if for $x \neq 0$ and y in $\mathscr{V}$ there exists A in $\mathfrak{A}$ with $Ax=y$. More generally, $\mathfrak{A}$ is *strictly n-fold transitive* if for any independent vectors $x_1, \ldots, x_n$ in $\mathscr{V}$ and arbitrary $y_1, \ldots, y_n$ in $\mathscr{V}$ there exists A in $\mathfrak{A}$ with $Ax_i = y_i$ for $i=1, \ldots, n$. If $\mathfrak{A}$ is strictly n-fold transitive for every n, then it is called *strictly dense*. The well-known theorem of JACOBSON [1] (see [2, p. 50]) states that 2-fold transitivity implies strict density. In general, 1-fold transitivity does not imply strict density.

Lemma 1. *Let $\mathfrak{A}$ be an algebra of bounded linear operators on the (not necessarily complete) complex normed linear space $\mathfrak{X}$. If $\mathfrak{A}$ is strictly transitive, then it is strictly dense.*

(Note that $\mathfrak{A}$ is not assumed to be closed in any topology.)

Proof. Suppose $\mathfrak{A}$ is strictly transitive but not strictly dense. Then by [1], $\mathfrak{A}$ is not 2-fold transitive. It follows (as in [2, p. 62]), that there exists a (not necessarily bounded) non scalar linear transformation T of $\mathfrak{X}$ onto $\mathfrak{X}$ that commutes with every A in $\mathfrak{A}$. Now

$$(T-\alpha I)A = A(T-\alpha I)$$

for every scalar α, and thus the nullspace and range of $T-\alpha I$ are invariant linear manifolds for $\mathfrak{A}$. It follows from the transitivity hypothesis that $T-\alpha I$ is bijective for every α. Thus $r(T)$ is a bijective linear transformation for every rational function r, and $r(T)A = Ar(T)$ for all A in $\mathfrak{A}$.

Fix a nonzero x_0 in $\mathfrak{X}$ and let $\mathfrak{X}_0$ be the linear manifold $\{r(T)x_0: r \text{ a rational function}\}$. Let $\mathfrak{A}_0 = \{A \in \mathfrak{A}: Ax_0 \in \mathfrak{X}_0\}$. Observe that $\mathfrak{X}_0$ is invariant under $\mathfrak{A}_0$. For each A in $\mathfrak{A}_0$ there is a rational function r_A such that $Ax_0 = r_A(T)x_0$ (r_A is unique because of the bijectivity of $r(T)$ for nonzero r); thus it follows from

$$Ar(T)x_0 = r(T)Ax_0 = r(T)r_A(T)x_0 = r_A(T)r(T)x_0$$

that the restriction of A to $\mathfrak{X}_0$ is just that of $r_A(T)$. Conversely, by the transitivity hypothesis, every $r(T)$ coincides with $r_A(T)$ for some A in $\mathfrak{A}_0$. Hence the restriction of $\mathfrak{A}_0$ to $\mathfrak{X}_0$ is a field. Since this restriction consists of bounded operators on $\mathfrak{X}_0$, the Gelfand—Mazur theorem implies that T is a scalar on $\mathfrak{X}_0$: $T \mid \mathfrak{X}_0 = \alpha I$, which contradicts the bijectivity of $T-\alpha I$ on $\mathfrak{X}$. This proves that $\mathfrak{A}$ is strictly dense.

Proof of Theorem 1. Let $\mathscr{V}$ be the intersection of all nonzero invariant linear manifolds of $\mathfrak{A}$. Then the restriction of $\mathfrak{A}$ to $\mathscr{V}$ is clearly strictly transitive and thus strictly dense. Also, $\mathfrak{A}$ has no closed invariant subspaces, because $\mathscr{V}$ is dense in

$\mathfrak{X}$. We use the notation and techniques described in [3, Chapter 8]. As in the proofs of Arveson's Lemma (Lemma 8.8) and Lemma 8.11 of [3], it suffices to show that each graph transformation for $\mathfrak{A}$ has an eigenvalue; this will imply the strong density of $\mathfrak{A}$ in $\mathscr{B}(\mathfrak{X})$.

Let $\{x\oplus T_1x\oplus\ldots\oplus T_nx\colon x\in\mathfrak{D}\}$ be an invariant graph subspace for $\mathfrak{A}^{(n+1)}$ for some positive integer n. We must show that each linear transformation T_i has an eigenvalue.

If T_i has nonzero null space, we are done. Otherwise, observe that the $\mathfrak{A}$-invariant linear manifolds $\mathfrak{D}$ and $T_i\mathfrak{D}$ both contain $\mathscr{V}$, by hypothesis. Hence the identities

$$AT_i = T_iA \quad \text{on } \mathfrak{D} \text{ and } \quad AT_i^{-1} = T_i^{-1}A \quad \text{on} \quad T_i\mathfrak{D}$$

(for all A in $\mathfrak{A}$) hold on $\mathscr{V}$. This implies that $T_i\mathscr{V}$ and $T_i^{-1}\mathscr{V}$ are also invariant under $\mathfrak{A}$, and thus contain $\mathscr{V}$. This yields $T_i\mathscr{V}=\mathscr{V}$. Now $\mathfrak{A}$ is strictly dense on $\mathscr{V}$ and commutes with the linear transformation T_i on $\mathscr{V}$. We conclude that T_i is a scalar on $\mathscr{V}$ and complete the proof.

We conclude with a question an affirmative answer to which would be a generalization of Theorem 1.

Question. If $\mathfrak{A}$ is a topologically transitive algebra of bounded linear operators on $\mathfrak{X}$ and if $\mathscr{L}(\mathfrak{A})$ has minimal nonzero elements, is $\mathfrak{A}$ necessarily strongly closed in $\mathscr{B}(\mathfrak{X})$?

References

[1] N. Jacobson, Structure theory of simple rings without finiteness assumptions, *Trans. Amer, Math. Soc.*, 57 (1944), 228—245.

[2] C. E. Rickart, *General Theory of Banach Algebras*, D. Van Nostrand (Princeton, 1960).

[3] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer-Verlag (Berlin—Heidelberg—New York, 1973).

DALHOUSIE UNIVERSITY
HALIFAX, N.S., CANADA

Acta Sci. Math., 47 (1984), 117—129

On the spectral residuum of closed operators

WANG SHENGWANG

1. Introduction

The spectral residuum [5], [2] of a linear operator T is a minimal closed subset S of the spectrum $\sigma(T)$, on whose complement T possesses the spectral properties of decomposable operators. It was shown in [2] that for every bounded linear operator there exists a spectral residuum. It is the purpose of the present paper to extend this property to the class of all closed operators which map a Banach space X into itself.

Throughout this paper, T denotes a closed operator with domain D_T and range in a complex Banach space X. $\mathbf{C}$ is the complex plane and $\mathbf{C}_\infty$ denotes its one-point compactification. All topological attributes for sets in $\mathbf{C}_\infty$ will be referred to the topology of $\mathbf{C}_\infty$. If $E\subset\mathbf{C}_\infty$, then $E^c=\mathbf{C}_\infty-E$ and $\bar{E}$ is the closure of E. For all operators involved in this paper, $\sigma(\cdot)$ denotes the extended spectrum. For a linear operator A, $\varrho(A)$ is the resolvent set and $R(\cdot\,;A)$ denotes the resolvent operator. Further notations will be given later.

We recall some basic concepts from [2], [5] and [6]. For $x\in X$ and $\lambda\in\mathbf{C}_\infty$, $\lambda\in\delta_T(x)$ if there exists a neighborhood U of λ and there is a function $f_x\colon U\to D_T$, analytic on U such that

$$(\mu-T)f_x(\mu)=x,\quad \mu\in U\cap\mathbf{C}.$$

Such a function f_x is said to be T-associated with x. Given T, there exists a unique maximal open set $\Omega_T\subset\mathbf{C}_\infty$ such that, for every set $G\subset\Omega_T$ and every analytic function f defined on G, the equation

$$(\mu-T)f(\mu)=0,\quad \mu\in G\cap\mathbf{C}$$

implies that $f(\mu)=0$ on G. Put $S_T=\Omega_T^c$ and for any $x\in X$, let

$$\gamma_T(x)=\delta_T(x)^c,\quad \sigma_T(x)=\gamma_T(x)\cup S_T,\quad \varrho_T(x)=\sigma_T(x)^c.$$

Received August 26, 1982.

Given T and $F \subset \mathbf{C}_\infty$, define the linear manifold

$$X_T(F) = \{x \in X: \ \sigma_T(x) \subset F\},$$

which is non void only if $F \supset S_T$ [5].

For a subspace (closed linear manifold) Y of X, we write $Y \in I(T)$ if $T(Y \cap D_T) \subset Y$ and $Y \in I_T$ if $Y \subset D_T$ and $T(Y) \subset Y$. For a closed $F \subset \mathbf{C}_\infty$, define

$$I(T, F) = \{Y \in I(T): \ \sigma(T|Y) \subset F\}, \quad I_{T,F} = \{Y \in I_T: \ \sigma(T|Y) \subset F\}.$$

The inclusion $\subset$ defines a partial ordering in the families $I(T, F)$ and $I_{T,F}$. If $I(T, F)$, $(I_{T,F})$ has an upper bound belonging to $I(T, F)$, $(I_{T,F})$, denote it by $X(T, F)$ (resp. $X_{T,F}$).

$Y \in I(T)$ is said to be a spectral maximal space of T if, for every $Z \in I(T)$, the relation $\sigma(T|Z) \subset \sigma(T|Y)$ implies $Z \subset Y$. It follows easily that if $F \subset \mathbf{C}_\infty$ is closed and $X(T, F)$ exists, then $X(T, F)$ is a spectral maximal space of T. Conversely, if Y is a spectral maximal space of T, then $Y = X(T, F)$, with $F = \sigma(T|Y)$.

Let $S \subset \mathbf{C}_\infty$ be closed and let n be a given positive integer. The family of open sets $\{G_S, G_1, \dots, G_n\}$ is called an (S, n)-covering of a closed set F, if

$$G_S \cup \left(\bigcup_{i=1}^{n} G_i \right) \supset F \cup S, \quad \bar{G}_i \cap S = \emptyset \quad \text{for} \quad i = 1, 2, \dots, n.$$

1.1. Definition. Given T, suppose $S \subset \sigma(T)$ is closed and n is a positive integer. T is called (S, n)-decomposable if, for any (S, n)-covering $\{G_S, G_1, G_2, \dots, G_n\}$ of $\sigma(T)$, there exist spectral maximal spaces $X_i \subset D_T$, $(i = 1, 2, \dots, n)$ and X_S of T, such that

$$X = X_S + \sum_{i=1}^{n} X_i, \quad \sigma(T|X_S) \subset \bar{G}_S, \quad \sigma(T|X_i) \subset \bar{G}_i \quad (i = 1, 2, \dots, n).$$

If T is (S, n)-decomposable for every positive integer n, then T is called S-decomposable.

Next, we list a few known properties that will be used in the subsequent theory.

1.2. Lemma. [3] *Given T, let F be closed such that $S_T \subset F \subset \mathbf{C}_\infty$. If $X_T(F)$ is closed, then $X_T(F) = X(T, F)$.*

1.3. Lemma. [3] *If T is $(S, 1)$-decomposable, then $S_T \subset S$.*

1.4. Lemma. [3] *If T is $(S, 1)$-decomposable and $F \supset S$ is closed, then*

$$X_T(F) = X(T, F) \quad \textit{and} \quad \sigma[T|X_T(F)] \subset F.$$

1.5. Lemma. [2, 7] *If T and $Y \in I(T)$ are such that $\sigma(T) \cup \sigma(T|Y) \neq \mathbf{C}$, then the coinduced operator $\hat{T}$ on the quotient space X/Y is closed and $\sigma(\hat{T}) \subset \sigma(T) \cup \sigma(T|Y)$, $\sigma(T|Y) \subset \sigma(T) \cup \sigma(\hat{T})$, $\sigma(T) \subset \sigma(\hat{T}) \cup \sigma(T|Y)$.*

1.6. Lemma. [8] *Given T, every spectral maximal space Y of T is hyperinvariant under T, in particular, $\sigma(T|Y)\subset\sigma(T)$.*

1.7. Theorem. [1] *Given T, for every $x\in X$ and $\lambda_0\in\mathbf{C}$, the following assertions are equivalent:*

(i) *there is a neighborhood $\delta\subset\mathbf{C}$ of λ_0 and there is a function $f\colon\delta\to D_T$, analytic on δ, satisfying*

$$(\lambda-T)f(\lambda)=x;$$

(ii) *there are numbers $M>0$, $R>0$ and a sequence $\{a_n\}_{n=0}^{\infty}\subset D_T$ with the following properties:*

(a) $(\lambda_0-T)a_0=x$; (b) $(\lambda_0-T)a_{n+1}=a_n$; (c) $\|a_n\|\leqq MR^n$ $(n=0,1,\ldots)$.

2. Some properties of $(S, 1)$-decomposable operators

2.1. Theorem. *Suppose that T is $(S,1)$-decomposable, H is closed in $\mathbf{C}_\infty$, $H\cap S=\emptyset$. Then $X_{T,H}$ exists and*

$$X_T(S\cup H)=X_T(S)\oplus X_{T,H}. \tag{2.1}$$

Proof. Put $F=S\cup H$. Lemma 1.4 implies that

$$X_T(F)=X(T,F)\quad\text{and}\quad\sigma[T|X_T(F)]\subset F=S\cup H.$$

Refer to [3, Theorem 1], consider $S_1=S_2=S$ in the hypotheses of Part (2) of the proof, note that the proof holds for $(S_i, 1)$-decomposable operators $(i=1,2)$ and conclude that $X_{T,H}$ exists and

$$X_T(F)=Z_S\oplus X_{T,H} \tag{2.2}$$

where $Z_S\in I(T)$ and $\sigma(T|Z_S)\subset S$. It remains to show that $Z_S=X_T(S)$. The existence of $X_T(S)$ follows from Lemma 1.4 and the inclusion $Z_S\subset X_T(S)$ is evident. Since $\sigma[T|X_T(S)]\subset S\subset F$, we have $X_T(S)\subset X_T(F)$. Letting $\sigma_H=\sigma(T|X_{T,H})$, it follows from [3, Theorem 1] that σ_H is bounded. Let D be a bounded Cauchy domain such that $\sigma_H\subset D\subset\bar{D}\subset S^c$, with the positively oriented boundary ∂D. Put

$$P_H=\frac{1}{2\pi i}\int_{\partial D}[\lambda-T|X_T(F)]^{-1}\,d\lambda.$$

It follows from $X_T(S)\subset X_T(F)$ and $\sigma[T|X_T(S)]\subset S$, that for every $x\in X_T(S)$, we have

$$P_Hx=\frac{1}{2\pi i}\int_{\partial D}[\lambda-T|X_T(F)]^{-1}x\,d\lambda=\frac{1}{2\pi i}\int_{\partial D}[\lambda-T|X_T(S)]^{-1}x\,d\lambda=0.$$

Therefore, $X_T(S) \subset Z_S$ and hence

$$X_T(S) = Z_S. \tag{2.3}$$

Relations (2.2) and (2.3) conclude the proof.

2.2. Remark. By the method used in the proof of Theorem 2.1, we can actually prove a more general result: If T is $(S, 1)$-decomposable and F, H are disjoint closed sets with $F \supset S$, then

$$X_T(F \cup H) = X_T(F) \oplus X_{T,H}.$$

2.3. Theorem. *Suppose that T is $(S, 1)$-decomposable and F, H are closed sets with $F \supset S$ and $S \cap H = \emptyset$. Then*

$$X_T(F) \cap X_{T,H} = X_{T,F\cap H}. \tag{2.4}$$

Proof. By Theorem 2.1, we have

$$X_T(S \cup H) = X_T(S) \oplus X_{T,H}, \quad X_T[S \cup (F \cap H)] = X_T(S) \oplus X_{T,F\cap H}.$$

Consequently,

$$\begin{aligned}[X_T(S) \oplus X_{T,H}] \cap X_T(F) = X_T(S \cup H) \cap X_T(F) = X_T[(S \cup H) \cap F] = \\ = X_T[S \cup (F \cap H)] = X_T(S) \oplus X_{T,F\cap H}.\end{aligned} \tag{2.5}$$

The following evident relations

$$X_T(S) + [X_T(F) \cap X_{T,H}] \subset X_T(F).$$

$$X_T(S) + [X_T(F) \cap X_{T,H}] \subset X_T(S) \oplus X_{T,H}$$

imply

$$X_T(S) + [X_T(F) \cap X_{T,H}] \subset [X_T(S) \oplus X_{T,H}] \cap X_T(F). \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$X_T(S) + [X_T(F) \cap X_{T,H}] \subset X_T(S) \oplus X_{T,F\cap H}. \tag{2.7}$$

Since, evidently

$$X_T(F) \cap X_{T,H} \supset X_{T,F\cap H}, \tag{2.8}$$

(2.7) is actually an equality. Moreover, the left-hand side of (2.7) being a direct sum, we obtain

$$X_T(S) \oplus [X_T(F) \cap X_{T,H}] = X_T(S) \oplus X_{T,F\cap H}. \tag{2.9}$$

Now, (2.8) and (2.9) imply that

$$X_T(F) \cap X_{T,H} = X_{T,F\cap H}$$

and hence (2.4) follows.

2.4. Theorem. *Suppose that T is $(S, 1)$-decomposable, and H_1, H_2 are closed disjoint sets such that $H_i \cap S = \emptyset$, $i=1,2$. Then*

$$X_{T,H_1 \cup H_2} = X_{T,H_1} \oplus X_{T,H_2}. \tag{2.10}$$

Proof. It follows from the relations

$$X_{T,H_1} \cap X_{T,H_2} \subset X_{T,H_1} \cap [X_T(S) \oplus X_{T,H_2}] = X_{T,H_1} \cap X_T(S \cup H_2)$$

and from Remark 2.2, that

$$X_{T,H_1} \cap X_{T,H_2} = \{0\}. \tag{2.11}$$

Since $X_{T,H_1 \cup H_2} \supset X_{T,H_i}$ $(i=1,2)$, (2.10) would follow from (2.11) if we could prove

$$X_{T,H_1 \cup H_2} \subset X_{T,H_1} + X_{T,H_2}. \tag{2.12}$$

Let $V = T|X_{T,H_1 \cup H_2}$. Then $\sigma(V) \subset H_1 \cup H_2$. Therefore, the sets $\sigma_{H_i} = \sigma(V) \cap H_i$ are disjoint spectral sets of V. It follows from [4, V. Theorem 9.2] that

$$X_{T,H_1 \cup H_2} = Z_{H_1} \oplus Z_{H_2} \quad \text{and} \quad \sigma(V|Z_{H_i}) = \sigma_{H_i} \quad (i = 1, 2).$$

Since V is bounded, $T|Z_{H_i} = V|Z_{H_i}$ are also bounded and then $Z_{H_i} \in I_{T,H_i}$ $(i=1,2)$. Hence $Z_{H_i} \subset X_{T,H_i}$ $(i=1,2)$ and (2.12) follows.

2.5. Theorem. *Suppose that T is $(S, 1)$-decomposable and H is a closed set satisfying $H \cap S = \emptyset$. Then $X(T, H)$ exists and $X(T, H) = X_{T,H}$.*

Proof. By Theorem 2.1, $X_{T,H}$ exists. If S is bounded then T is bounded [6, Proposition 3.1] and the statement of the theorem is evident. So suppose that $\infty \in S$. Then $H \cap S = \emptyset$ implies that H is bounded. As we mentioned in the Introduction, for every operator appearing in this paper, we consider the extended spectrum. Hence, for each $Y \in I(T, H)$, $\sigma(T|Y) \subset H$ implies that the extended spectrum $\sigma(T|Y)$ is bounded. Then $Y \in I_{T,H}$ and hence $I(T, H) \subset I_{T,H}$. On the other hand, we evidently have $I_{T,H} \subset I(T, H)$. Thus,

$$I(T, H) = I_{T,H} \tag{2.13}$$

and the conclusion of the proof follows immediately from (2.13).

2.6. Theorem. *Suppose that T is $(S, 1)$-decomposable and G is open in $\mathbf{C}_\infty$ such that $\bar{G} \cap S = \emptyset$. Then the coinduced operator $T^{\bar{G}}$ on the quotient space $X/X_{T,\bar{G}}$ is closed and $\sigma(T^{\bar{G}}) \subset G^c$.*

Proof. Let $\lambda \in G$ and let $G_S \supset S$ be open in $\mathbf{C}_\infty$ such that $\{G_S, G\}$ is an $(S, 1)$-covering of $\sigma(T)$ and $\lambda \notin \bar{G}_S$. By Lemma 1.4 and Theorem 2.5, $X_T(\bar{G}_S)$ and $X_{T,\bar{G}}$ are spectral maximal spaces of T. Consequently,

$$X = X_T(\bar{G}_S) + X_{T,\bar{G}}. \tag{2.14}$$

Let $\cong$ denote the topological isomorphism between two Banach spaces. In view of (2.14),

$$X/X_{T,\bar{G}} \cong X_T(\bar{G}_S)/X_T(\bar{G}_S) \cap X_{T,\bar{G}}.$$

It follows from Theorem 2.3 that

$$X_T(\bar{G}_S) \cap X_{T,\bar{G}} = X_{T,\bar{G}_S \cap \bar{G}}$$

and hence

$$X/X_{T,\bar{G}} \cong X_T(\bar{G}_S)/X_{T,\bar{G}_S \cap \bar{G}}. \tag{2.15}$$

In view of (2.15), $T^{\bar{G}}$ can be considered as an operator on $X_T(\bar{G}_S)/X_{T,\bar{G}_S \cap \bar{G}}$. Since $\lambda \notin \bar{G}_S$ and $\sigma[T|X_T(\bar{G}_S)] \cup \sigma(T|X_{T,\bar{G}_S \cap \bar{G}}) \subset \bar{G}_S$, it follows from Lemma 1.5 that $T^{\bar{G}}$ is closed and $\lambda \notin (T^{\bar{G}})$. As λ is arbitrary in G, we have $\sigma(T^{\bar{G}}) \subset G^c$.

3. Equivalence of closed $(S, 1)$-decomposable and S-decomposable operators

3.1. Theorem. *Suppose that T is $(S, 1)$-decomposable and $G \subset \mathbf{C}$ is open in $\mathbf{C}_\infty$ such that $\bar{G} \cap S = \emptyset$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of analytic D_T-valued functions defined on G, with the property*

$$h_n(\lambda) = (\lambda - T)f_n(\lambda) \to 0 \quad as \quad n \to \infty \tag{3.1}$$

in the strong topology of X and uniformly on every bounded subset of G. Then

$$f_n(\lambda) \to 0 \quad as \quad n \to \infty$$

in the strong topology of X and uniformly on every bounded subset of G.

Proof. We may suppose that

$$G = \{\lambda \in \mathbf{C}\colon\ |\lambda| < R,\ R > 0\}.$$

By decreasing R, we may suppose that (3.1) holds uniformly on G. Let R_0 with $0 < R_0 < R$ be arbitrary. Choose the numbers R_1, R_1', R_2', R_2 such that $R_0 < R_1 < R_1' < R_2' < R_2 < R$ and put

$$G_j = \{\lambda \in \mathbf{C}\colon\ |\lambda| < R_j\}, \quad j = 0, 1;$$

$$H = \{\lambda \in \mathbf{C}\colon\ R_1 \leqq |\lambda| \leqq R_2\}; \quad H' = \{\lambda \in \mathbf{C}\colon\ R_1' \leqq |\lambda| \leqq R_2'\}.$$

By Theorem 2.6, the coinduced operator T^H on $X/X_{T,H}$ is closed and

$$\sigma(T^H) \subset (H^0)^c, \tag{3.2}$$

where $H^0 = \{\lambda \in \mathbf{C}\colon\ R_1 < |\lambda| < R_2\}$.

If $x \in X$ and f is an X-valued function, then we use the notations $\hat{x} = x + X_{T,H}$ and $\hat{f}(\lambda) = f(\lambda) + X_{T,H}$ for the cosets in the quotient space $X/X_{T,H}$. In $\hat{X}/X_{T,H}$, the convergence (3.1) gives rise to

$$\hat{h}_n(\lambda) = (\lambda - T^H)\hat{f}_n(\lambda) \to 0 \quad (n \to \infty)$$

in the strong topology of $X/X_{T,H}$ and uniformly on G. In view of (3.2), $(\lambda - T^H)^{-1}$ is uniformly bounded on H' and hence

$$\hat{f}_n(\lambda) = (\lambda - T^H)^{-1}\hat{h}_n(\lambda) \to 0 \quad (n \to \infty)$$

in the strong topology of $X/X_{T,H}$ and uniformly on H'. By the maximum principle,

$$\hat{f}_n(\lambda) \to 0 \quad (n \to \infty)$$

in the strong topology of $X/X_{T,H}$ and uniformly on $\bar{G}_1$.

For $\lambda \in G$ and $n = 1, 2, \ldots,$ let

$$f_n(\lambda) = \sum_{k=0}^{\infty} a_{nk}\lambda^k$$

be the power series expansion of f_n. Then

$$\hat{f}_n(\lambda) = \sum_{k=0}^{\infty} \hat{a}_{nk}\lambda^k.$$

By the Cauchy inequalities, we have

$$\|\hat{a}_{nk}\| \leqq \varepsilon_n / R_1^k, \quad n = 1, 2, \ldots, \quad k = 0, 1, \ldots,$$

where

$$\varepsilon_n = \max\{\|\hat{f}_n(\lambda)\| : \lambda \in \bar{G}_1\} \to 0 \quad (n \to \infty).$$

For every $\hat{a}_{nk}$, there is $b_{nk} \in \hat{a}_{nk}$ such that $\|b_{nk}\| \leqq 2\|\hat{a}_{nk}\|$. For every n, let

(3.3) $$g_n(\lambda) = \sum_{k=0}^{\infty} b_{nk}\lambda^k.$$

Then

$$\|g_n(\lambda)\| \leqq \sum_{k=0}^{\infty} \|b_{nk}\| \cdot |\lambda|^k \leqq 2\varepsilon_n \sum_{k=0}^{\infty} |\lambda|^k / R_1^k, \quad \lambda \in G_1$$

and hence the series (3.3) is absolutely and uniformly convergent in $\bar{G}_0$, with

(3.4) $$\|g_n(\lambda)\| \leqq 2\varepsilon_n R_1/(R_1 - R_0) \to 0, \quad \lambda \in \bar{G}_0.$$

Since $b_{nk} \in \hat{a}_{nk}$ implies that $\hat{f}_n(\lambda) = \hat{g}_n(\lambda)$ on $\bar{G}_0$, we have

(3.5) $$k_n(\lambda) = f_n(\lambda) - g_n(\lambda) \in X_{T,H}, \quad \lambda \in \bar{G}_0.$$

Next, consider positive numbers $\tilde{R}_1, \tilde{R}_1', \tilde{R}_2', \tilde{R}_2$ related by the inequalities $R_0 < \tilde{R}_1 < \tilde{R}_1' < \tilde{R}_2' < \tilde{R}_2 < R_1$ and put $\tilde{H} = \{\lambda \in \mathbf{C} : \tilde{R}_1 \leqq |\lambda| \leqq \tilde{R}_2\}$. All the above conclusions remain valid for $\tilde{R}_1, \tilde{R}_1', \tilde{R}_2', \tilde{R}_2$ substituting R_1, R_1', R_2', R_2, respectively. Hence, for $n=1, 2, \ldots,$ there exists an X-valued analytic function $\tilde{g}_n$ with

$$\|\tilde{g}_n(\lambda)\| \leqq 2\tilde{\varepsilon}_n\tilde{R}_1/(\tilde{R}_1-R_0) \to 0, \quad \lambda\in\bar{G}_0, \tag{3.6}$$

where $\tilde{\varepsilon}_n$ is the analogue of ε_n. Furthermore, we have

$$\tilde{k}_n(\lambda) = f_n(\lambda)-\tilde{g}_n(\lambda)\in X_{T,\tilde{H}}, \quad \lambda\in\bar{G}_0. \tag{3.7}$$

Now, subtract (3.7) from (3.5) and use (3.4) and (3.6) to obtain

$$\|k_n(\lambda)-\tilde{k}_n(\lambda)\| = \|g_n(\lambda)-\tilde{g}_n(\lambda)\| \leqq 2(\varepsilon_n+\tilde{\varepsilon}_n)\tilde{R}_1/(\tilde{R}_1-R_0) \to 0; \quad \lambda\in\bar{G}_0. \tag{3.8}$$

Since H and $\tilde{H}$ are disjoint bounded closed sets with $S\cap H=\emptyset$, $S\cap\tilde{H}=\emptyset$, Theorem 2.4 implies that

$$X_{T,H\cup\tilde{H}} = X_{T,H}\oplus X_{T,\tilde{H}}.$$

Hence, there is $M>0$ so that, for $x\in X_{T,H}$ and $\tilde{x}\in X_{T,\tilde{H}}$,

$$\|x\|+\|\tilde{x}\| \leqq M\|x+\tilde{x}\|. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\|k_n(\lambda)\| \leqq 2(\varepsilon_n+\tilde{\varepsilon}_n)M\tilde{R}_1/(\tilde{R}_1-R_0) \to 0, \quad \lambda\in\bar{G}_0. \tag{3.10}$$

Thus, (3.5), (3.4) and (3.10) imply that

$$\|f_n(\lambda)\| \leqq \|k_n(\lambda)\|+\|g_n(\lambda)\| \to 0$$

uniformly on $\bar{G}_0$. Since $R_0\in(0, R)$ is arbitrary, the proof is complete.

It is easily seen that if $\{f_n\}$ in Theorem 3.1 is replaced by a double sequence, then the conclusion remains valid.

3.2. Corollary. *Suppose that T is $(S, 1)$-decomposable, $G\subset\mathbf{C}$ is open in $\mathbf{C}_\infty$ such that $\bar{G}\cap S=\emptyset$. If $\{f_{nm}\colon G\to D_T\}$ is a double sequence of functions, analytic on G such that*

$$(\lambda-T)f_{nm}(\lambda) \to 0 \quad (n, m \to\infty)$$

in the strong topology of X and uniformly on every bounded subset of G, then

$$f_{nm}(\lambda) \to 0 \quad (n, m \to\infty)$$

in the strong topology of X and uniformly on every bounded subset of G.

3.3. Theorem. *Let T be $(S, 1)$-decomposable. If for $x\in X$ there is a sequence $\{f_n\colon G\to D_T\}$ of analytic functions on an open set $G\subset\mathbf{C}$ with $\bar{G}\cap S=\emptyset$, such that*

$$\|x-(\lambda-T)f_n(\lambda)\| \to 0 \quad (n \to\infty) \tag{3.11}$$

uniformly on every bounded subset of G, then $G\subset\varrho_T(x)$.

Proof. Put $f_{nm}(\lambda)=f_n(\lambda)-f_m(\lambda)$, $\lambda\in G$. Corollary 3.2 implies that $f_{nm}(\lambda)\to 0$ $(n, m\to\infty)$ in the strong topology of X and uniformly on every bounded subset of G. Then the function $f\colon G\to X$, defined by

$$f(\lambda) = \lim_{n\to\infty} f_n(\lambda)$$

is analytic on G. Since T is closed, (3.11) implies that

$$f(\lambda)\in D_T \quad\text{and}\quad (\lambda-T)f(\lambda) = x \quad\text{for}\quad \lambda\in G. \tag{3.12}$$

Since, by Lemma 1.3, $\bar{G}\cap S_T\subset\bar{G}\cap S=\emptyset$, (3.12) implies that $G\subset\sigma_T(x)$.

3.4. Theorem. *Suppose that T is $(S, 1)$-decomposable, $F\subset\mathbf{C}_\infty$ is closed such that $X(T, F)$ (resp. $X_{T,F}$) exists. Then for every (S, m)-covering $\{G_S, G_1, \ldots, G_m\}$ of F, where m is a positive integer, we have*

$$X(T, F) \subset X_T(\bar{G}_S)+\sum_{i=1}^{m} X_{T,\bar{G}_i}, \tag{3.13}$$

respectively,

$$X_{T,F} \subset X_T(\bar{G}_S)+\sum_{i=1}^{m} X_{T,\bar{G}_i}. \tag{3.13'}$$

Proof. We confine the proof to (3.13), that of (3.13′) being similar.

If S is bounded, the statement of the theorem is [2, Theorem 4]. Therefore, we suppose that $\infty\in S$. We divide the proof into four parts.

Part A. Assume that $m=1$. Then $\{G_S, G_1\}$ is an $(S, 1)$-covering of F. Let $H=\overline{G_S\cap G_1}$. Then $H\cap S=\emptyset$ and by Theorems 2.1 and 2.6, $X_{T,H}$ exists, the coinduced operator T^H on $X/X_{T,H}$ is closed and

$$\sigma(T^H) \subset (G_S\cap G_1)^c. \tag{3.14}$$

For the cosets in $X/X_{T,H}$ and for the $X/X_{T,H}$-valued functions we use the notations introduced in the proof of Theorem 3.1.

Let $x\in X(T, F)$ and put $x(\lambda)=[\lambda-T|X(T, F)]^{-1}x$, for $\lambda\in F^c$. It follows from $(\lambda-T)x(\lambda)=x$, that $(\lambda-T^H)\hat{x}(\lambda)=\hat{x}$, $\lambda\in F^c$. In view of (3.14), the resolvent operator $R(\lambda; T^H)$ is defined for $\lambda\in G_S\cap G_1$. Define

$$\hat{f}(\lambda) = \begin{cases} \hat{x}(\lambda), & \text{if} \quad \lambda\in F^c,\\ R(\lambda; T^H)\hat{x}, & \text{if} \quad \lambda\in G_S\cap G_1. \end{cases}$$

Clearly, $\hat{f}$ is well-defined and is analytic on $F^c\cup(G_S\cap G_1)$. Since $\infty\in S\subset G_S$, $F-G_S$ is bounded. Let D be a bounded Cauchy domain such that $F-G_S\subset D$ and $\bar{D}\cap(F-G_1)=\emptyset$. If ∂D is the positively oriented boundary of D, put

$$\hat{x}_0 = \frac{1}{2\pi i}\int_{\partial D}\hat{f}(\lambda)\,d\lambda, \quad \hat{x}_1 = \hat{x}-\hat{x}_0. \tag{3.15}$$

Evidently, $\hat{x}_0$ is independent of the choice of D. Now (3.15) gives rise to the following representation of x:

$$(3.16) \qquad x = x_0 + x_1 + y, \quad \text{with} \quad x_i \in \hat{x}_i \quad (i = 0, 1), \quad y \in X_{T,H}.$$

Part B. In this part we prove that $x_0 \in X_T(\bar{G}_S) + X_{T,\bar{G}_1}$.

Let $\lambda_0 \notin S \cup \bar{G}_1$ and let δ be a neighborhood of λ_0 so that $\bar{\delta} \cap (S \cup \bar{G}_1) = \emptyset$. We may choose the Cauchy domain D satisfying $\bar{D} \cap \bar{\delta} = \emptyset$. For $\lambda \in \delta$, we have successively

$$(\lambda - T^H) \frac{1}{2\pi i} \int_{\partial D} \frac{\hat{f}(\mu)}{\lambda - \mu} d\mu = \frac{1}{2\pi i} \int_{\partial D} \frac{(\lambda - T^H)\hat{f}(\mu)}{\lambda - \mu} d\mu =$$

$$= \frac{1}{2\pi i} \int_{\partial D} \frac{\hat{x}}{\lambda - \mu} d\mu + \frac{1}{2\pi i} \int_{\partial D} \hat{f}(\mu)\, d\mu = \hat{x}_0.$$

By Theorem 1.7, there is a sequence $\{\hat{a}_n\}_{n=0}^{\infty} \subset D_{TH}$ and there are numbers $M > 0$, $R > 0$, such that

$$(3.17) \quad (\lambda_0 - T^H)\hat{a}_0 = \hat{x}_0, \quad (\lambda_0 - T^H)\hat{a}_{n+1} = \hat{a}_n, \quad \|\hat{a}_n\| \leqq MR^n, \quad n = 0, 1, \ldots.$$

By the definition of D_{TH}, $\hat{a}_n \cap D_T \neq \emptyset$. Let $a_n \in \hat{a}_n \cap D_T$. Then $\hat{a}_n = a_n + X_{T,H} \subset D_T$ and hence we may choose a_n to satisfy the inequality $\|a_n\| \leqq 2\|\hat{a}_n\|$, $n = 0, 1, \ldots$. In view of (3.17), we have

$$(3.18) \qquad (\lambda_0 - T)a_0 = x_0 + b_0, \quad (\lambda_0 - T)a_{n+1} = a_n + b_{n+1},$$
$$\|a_n\| \leqq 2MR^n, \quad n = 0, 1, \ldots$$

where $\{b_n\}_{n=0}^{\infty} \subset X_{T,H}$. Let

$$A_n(\lambda) = \sum_{k=0}^{n} a_k (\lambda_0 - \lambda)^k, \quad B_n(\lambda) = \sum_{k=0}^{n} b_k (\lambda_0 - \lambda)^k.$$

Then, it follows from

$$\sigma(T|X_{T,H}) \cap \bar{\delta} \subset H \cap \bar{\delta} \subset \bar{G}_1 \cap \bar{\delta} = \emptyset,$$

that for $\lambda \in \delta$,

$$(\lambda - T)[A_n(\lambda) - (\lambda - T|X_{T,H})^{-1} B_n(\lambda)] = (\lambda - T)A_n(\lambda) - B_n(\lambda) =$$
$$= x_0 - a_n(\lambda_0 - \lambda)^{n+1}.$$

Let $\delta_0 = \{\lambda \in \mathbf{C}\colon |\lambda - \lambda_0| < 1/2R\}$. For $\lambda \in \delta \cap \delta_0$, the last inequality of (3.18), implies

$$\|a_n\| \cdot |\lambda_0 - \lambda|^{n+1} \leqq M/2^n R \to 0 \quad (n \to \infty)$$

and hence

$$\|(\lambda - T)[A_n(\lambda) - (\lambda - T|X_{T,H})^{-1} B_n(\lambda)] - x_0\| \to 0,$$

uniformly on $\delta \cap \delta_0$. By Theorem 3.3, $\delta \cap \delta_0 \subset \varrho_T(x_0)$ and hence $\lambda_0 \in \varrho_T(x_0)$. Since

$\lambda_0 \notin S \cup \bar{G}_1$ is arbitrary, we have $\sigma_T(x_0) \subset S \cup \bar{G}_1$. Thus,

$$x_0 \in X_T(S \cup \bar{G}_1) = X_T(S) \oplus X_{T,\bar{G}_1} \subset X_T(\bar{G}_S) + X_{T,\bar{G}_1}. \tag{3.19}$$

Part C. In this part we show that $x_1 \in X_T(\bar{G}_S)$. Let $\lambda_0 \notin \bar{G}_S$. There exists a neighborhood γ of λ_0 such that $\bar{\gamma} \cap \bar{G}_S = \emptyset$. We can choose a Cauchy domain D such that $D \supset \bar{\gamma} \cup (F - G_S)$. Then for $\lambda \in \gamma$, we obtain successively

$$(\lambda - T^H)\left[-\frac{1}{2\pi i}\int_{\partial D}\frac{\hat{f}(\mu)}{\lambda-\mu}\,d\mu\right] = -\frac{1}{2\pi i}\int_{\partial D}\frac{(\lambda - T^H)\hat{f}(\mu)}{\lambda-\mu}\,d\mu =$$

$$= -\frac{1}{2\pi i}\int_{\partial D}\hat{f}(\mu)\,d\mu + \frac{1}{2\pi i}\int_{\partial D}\frac{\hat{x}}{\mu-\lambda}\,d\mu = \hat{x} - \hat{x}_0 = \hat{x}_1.$$

By repeating the method used in Part B, one obtains

$$x_1 \in X_T(\bar{G}_S). \tag{3.20}$$

Part D. It follows from (3.16), (3.19), (3.20) and $y \in X_{T,H} \subset X_T(\bar{G}_S)$, that

$$X(T, F) \subset X_T(\bar{G}_S) + X_{T,\bar{G}_1}.$$

A subsequent repetition, via induction on m, leads one to (3.13).

3.5. Theorem. *Every closed* $(S, 1)$*-decomposable operator is* S*-decomposable.*

Proof. Let $\{G_S, G_1, \ldots, G_n\}$ be an S-covering of $\sigma(T)$. By Theorem 3.4, we have

$$X = X[T, \sigma(T)] \subset X_T(\bar{G}_S) + \sum_{i=1}^{n} X_{T,\bar{G}_i} \subset X$$

and hence T is S-decomposable.

4. The spectral residuum

4.1. Definition. Given T, let $\Sigma(T)$ be the family of all closed sets S such that $S_T \subset S \subset \sigma(T)$ and T is S-decomposable. If there exists $S^* \in \Sigma(T)$ such that $S^* \subset S$ for any $S \in \Sigma(T)$, then S^* is called the spectral residuum of T.

4.2. Theorem. *The spectral residuum exists for every closed operator* T.

Proof. We only sketch the proof because it is similar to that of [2, Theorem 6]. Since $\sigma(T)$ is in $\Sigma(T)$, $\Sigma(T)$ is nonempty. Let $\{S_\alpha\colon \alpha \in A\}$ be a totally ordered subfamily of $\Sigma(T)$ and let $S_0 = \bigcap \{S_\alpha\colon \alpha \in A\}$. If $H \subset \mathbf{C}_\infty$ is a closed set disjoint from S_0 then, since $\mathbf{C}_\infty$ is compact, there is $\alpha \in A$ such that $H \cap S_\alpha = \emptyset$. Hence an S_0-covering of $\sigma(T)$ is an S_α-covering of $\sigma(T)$ for some $\alpha \in A$. Since T is S_α-decomposable, it is also S_0-decomposable. By Zorn's lemma, there is a minimal element in $\Sigma(T)$. It remains to prove that, for $S_1, S_2 \in \Sigma(T)$, $S = S_1 \cap S_2 \in \Sigma(T)$.

Let $\{G_S, G\}$ be an S-covering of $\sigma(T)$. In view of [3, Theorem 1 (6)] or [2, Theorem 6], we may choose open sets G_{S_i}, G_i $(i=1, 2)$, such that

$$G_{S_i} \supset S_i \cup G_S, \quad i = 1, 2; \tag{4.1}$$

$$\bar{G}_{S_1} \cap \bar{G}_{S_2} = \bar{G}_S, \tag{4.2}$$

$$G_i \subset G, \quad \bar{G}_i \cap S_i = \emptyset, \quad G_i \cup G_{S_i} \supset G, \quad i = 1, 2. \tag{4.3}$$

Thus, $\{G_{S_i}, G_i\}$ $(i=1, 2)$ is an $(S_i, 1)$-covering of $\sigma(T)$. Let G'_{S_2} be open in $\mathbf{C}_\infty$ such that $\bar{G}'_{S_2} \subset G_{S_2}$ and $\{G'_{S_2}, G_2\}$ is an $(S_2, 1)$-covering of $\sigma(T)$. Since T is S_2-decomposable, we have

$$X = X_T(\bar{G}'_{S_2}) + X_{T, \bar{G}_2}. \tag{4.4}$$

Since T is S_i-decomposable $(i=1, 2)$, $X_{T, \bar{G}}$ exists by part 2 of the proof of [3, Theorem 1]. It follows from $G_2 \subset G$ and (4.4), that

$$X = X_T(\bar{G}'_{S_2}) + X_{T, \bar{G}}. \tag{4.5}$$

Put $F = \bar{G}'_{S_2} \cap \sigma(T)$. Since $X_T(\bar{G}'_{S_2})$ is a spectral maximal space of T, by Lemma 1.6,

$$\sigma[T|X_T(\bar{G}'_{S_2})] \subset \sigma(T).$$

Thus, we have

$$\sigma[T|X_T(\bar{G}'_{S_2})] \subset \bar{G}'_{S_2} \cap \sigma(T) = F$$

and it follows easily that $X_T(\bar{G}'_{S_2})$ is the upper bound of $I(T, F)$, i.e.

$$X_T(\bar{G}'_{S_2}) = X(T, F). \tag{4.6}$$

Furthermore, $S_T \subset S_1 \cap S_2 = S \subset \bar{G}_S$ and (4.2) imply that $X_T(\bar{G}_S)$ exists and

$$X_T(\bar{G}_S) = X_T(\bar{G}_{S_1}) \cap X_T(\bar{G}_{S_2}).$$

Hence $X_T(\bar{G}_S)$ is closed. Similarly, $S = S_1 \cap S_2$ implies that

$$X_T(S) = X_T(S_1) \cap X_T(S_2)$$

and hence $X_T(S)$ is closed.

By (4.2), we have $G_{S_1} \cap G_{S_2} \subset G_S$, and hence

$$F = \bar{G}_{S_2} \cap \sigma(T) \subset G_{S_2} \cap (G_{S_1} \cup G_1) \subset (G_{S_2} \cap G_{S_1}) \cup G_1 \subset G_S \cup G_1.$$

Next, we prove

$$X(T, F) \subset X_T(\bar{G}_S) + X_{T, \bar{G}_1}. \tag{4.7}$$

Let $H = \overline{G_S \cap G_1}$. Then $H \cap S_1 \subset \bar{G}_1 \cap S_1 = \emptyset$. Since T is S_1-decomposable, $X_{T,H}$ exists by Theorem 2.1, the coinduced operator T^H on $X/X_{T,H}$ is closed and $\sigma(T^H) \subset$ $\subset (G_S \cap G_1)^c$ by Theorem 2.6. By repeating parts A, B and C of the proof of Theorem 3.4, one obtains that, for every $x\in X(T, F)$, $x=x_0+x_1+y$, where $y\in X_{T,H}$, $\sigma_T(x_1)\subset \bar{G}_S$ and $\sigma_T(x_0)\subset S\cup\bar{G}_1$. Hence

$$y\in X_{T,H}\subset X_T(\bar{G}_S), \tag{4.8}$$

$$x_1\in X_T(\bar{G}_S). \tag{4.9}$$

As for x_0, by repeating the proof of Theorem 2.1, we obtain

$$x_0\in X_T(S)\oplus X_{T,\bar{G}_1}\subset X_T(\bar{G}_S)+X_{T,\bar{G}_1}. \tag{4.10}$$

Thus (4.7) follows from (4.8), (4.9) and (4.10). In view of (4.3), we have $X_{T,\bar{G}_1}\subset X_{T,\bar{G}}$ and then, with the help of (4.5), (4.6), (4.7), we obtain

$$X=X_T(\bar{G}_S)+X_{T,\bar{G}}.$$

Thus, T is $(S, 1)$-decomposable. Theorem 3.5 concludes the proof.

Acknowledgement. The author would like to express his gratitude to Professor I. Erdélyi for his help.

References

[1] I. Erdélyi and Wang Shengwang, A characterization of decomposable operators, to appear.
[2] B. Nagy, On S-decomposable operators, *J. Operator Theory,* **2** (1979), 277—286.
[3] B. Nagy, A strong spectral residuum for every closed operator, *Illinois J. Math.,* **24** (1980), 173—179.
[4] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis,* Wiley (New York, 1980).
[5] F.-H. Vasilescu, Residually decomposable operators in Banach spaces, *Tôhoku Math. J.,* **21** (1969), 509—522.
[6] F.-H. Vasilescu, Residual properties for closed operators on Fréchet spaces, *Illinois J. Math.,* **15** (1971), 377—386.
[7] Wang Shengwang and I. Erdélyi, Analytically invariant spectral resolvents of closed operators, to appear.
[8] Wang Shengwang and I. Erdélyi, A spectral duality theorem for closed operators, to appear.

DEPARTMENT OF MATHEMATICS
NANJING UNIVERSITY
NANJING, CHINA

Acta Sci. Math., 47 (1984), 131—146

On commuting unbounded self-adjoint operators. I

KONRAD SCHMÜDGEN

Dedicated to Professor B. Szőkefalvi-Nagy on the occasion of his 70th birthday

Let A and B be unbounded self-adjoint operators in a Hilbert space $\mathscr{H}$ which are both essentially self-adjoint on a common dense domain $\mathscr{D}\subseteq\mathscr{D}(AB)\cap\mathscr{D}(BA)$ in $\mathscr{H}$ and commute on $\mathscr{D}$. We then write $\{A, B\}\in\mathfrak{N}_1$. It is well known that the spectral projections of A and B may fail to commute for $\{A, B\}\in\mathfrak{N}_1$. The first counter-example was constructed by NELSON [10]; see also [6], [9], [13], [15]. In this paper we begin a study of this phenomenon in terms of commutators of bounded operators. In the present paper we restrict ourselves to the case where the spectra $\sigma(A)$ and $\sigma(B)$ are both different from the real line. A similar approach is possible in the general case if we use the Cayley transforms of A and B. But the methods of construction are somewhat different in that case (we have to deal with commutators of two unitaries).

Suppose that $\alpha\in\mathbf{R}_1\setminus\sigma(A)$ and $\beta\in\mathbf{R}_1\setminus\sigma(B)$. In Section 1 we characterize the couples $\{A, B\}$ in $\mathfrak{N}_1$ in terms of the bounded self-adjoint operators $X:=(A-\alpha)^{-1}$ and $Y:=(B-\beta)^{-1}$. We show that $\{A, B\}\in\mathfrak{N}_1$ if and only if $\overline{\mathscr{R}([X, Y])}\cap\mathscr{R}(X)=$ $=\overline{\mathscr{R}([X, Y])}\cap\mathscr{R}(Y)=\{0\}$. Probably the simplest example of this kind for which $[X, Y]\neq 0$ is given by $X=\mathrm{Re}\, S$, $Y=\mathrm{Im}\, S$, where S is the unilateral shift. Therefore, $\{A:=(\mathrm{Re}\, S)^{-1}, B:=(\mathrm{Im}\, S)^{-1}\}\in\mathfrak{N}_1$, but A and B do not commute strongly.

In the remaining sections of the paper we establish pairs of bounded self-adjoint operators X, Y having these properties. We describe three typical situations. All irreducible pairs in $\mathfrak{N}_1$ for which the commutator $[X, Y]$ has rank one are classified in Section 2. Here we use the principal function [11] of the pair X, Y and the tracial bilinear form [8]. Toeplitz operators (mainly with polynomial symbols) are considered in Section 3. In Section 4 we study pairs of the class $\mathfrak{N}_1$ obtained by taking real and imaginary parts of certain one-dimensional "perturbations" of normal operators.

Received November 16, 1982.

Let us fix some notation. If T is an operator in a Hilbert space $\mathscr{H}$, then we use $\mathscr{D}(T)$, $\mathscr{N}(T)$, $\mathscr{R}(T)$ and $\sigma(T)$ to denote the domain, the kernel, the range and the spectrum of T, respectively. For a subset $\mathscr{K}$ of $\mathscr{H}$, $\overline{\mathscr{K}}$ is the closure of $\mathscr{K}$ in the Hilbert space norm. We denote by $\mathbf{N}_0$ and $\mathbf{N}$ the non-negative, resp., positive integers.

1. The class $\mathfrak{N}_1$

Throughout this section, let A and B denote self-adjoint operators in a Hilbert space $\mathscr{H}$.

1.1. Definition 1. We say that the couple $\{A, B\}$ is of the class $\mathfrak{N}_1$ if there exists a linear subspace $\mathscr{D}$ of $\mathscr{H}$ such that

(1) $\mathscr{D} \subseteq \mathscr{D}(AB) \cap \mathscr{D}(BA)$ and $AB\varphi = BA\varphi$ for all $\varphi \in \mathscr{D}$.

(2) $\mathscr{D}$ is dense in $\mathscr{H}$.

(3) $A \upharpoonright \mathscr{D}$ and $B \upharpoonright \mathscr{D}$ are essentially self-adjoint*) (e.s.a.).

Remarks. 1. Suppose that $\{A, B\} \in \mathfrak{N}_1$. If A (or B) is bounded, then A and B *strongly commute* (that is, by definition, the spectral projections $E(\lambda)$ of A and $F(\mu)$ of B commute for all $\lambda, \mu \in \mathbf{R}_1$). We sketch the proof. Since A is bounded and $B \upharpoonright \mathscr{D}$ is e.s.a., (1) extends by continuity on $\mathscr{D}(B)$, i.e., $AB\varphi = BA\varphi$ for all $\varphi \in \mathscr{D}(B)$. Since B is self-adjoint, this gives $[A, F(\mu)] = 0$ for $\mu \in \mathbf{R}_1$ and hence $[E(\lambda), F(\mu)] = 0$ for $\lambda, \mu \in \mathbf{R}_1$.

2. A pair $\{A, B\}$ in $\mathfrak{N}_1$ is said to be *irreducible* if each decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, where A_j and B_j are self-adjoint operators in the Hilbert spaces $\mathscr{H}_j$, $j = 1, 2$, is trivial, that is, $\mathscr{H}_1 = \{0\}$ or $\mathscr{H}_2 = \{0\}$. Obviously, this is the case if and only if each projection commuting with A and B is either 0 or I.

1.2. As mentioned above, we restrict ourselves in this paper to the case where $\sigma(A) \neq \mathbf{R}_1$ and $\sigma(B) \neq \mathbf{R}_1$. Suppose that $\alpha \in \mathbf{R}_1 \setminus \sigma(A)$ and $\beta \in \mathbf{R}_1 \setminus \sigma(B)$. We now reformulate the conditions occurring in Definition 1 in terms of the bounded self-adjoint operators $X := (A - \alpha)^{-1}$ and $Y := (B - \beta)^{-1}$.

For let P denote the orthogonal projection of $\mathscr{H}$ on $\overline{\mathscr{R}([X, Y])}$ and let $\mathscr{D}(A, B) := XY(I - P)\mathscr{H}$.

Definition 2. If $\{A, B\} \in \mathfrak{N}_1$, then $d(A, B) := \dim P\mathscr{H}$ is called the defect number of the pair $\{A, B\}$.

---

*) Recall that a symmetric operator T is called essentially selfadjoint if its closure $\overline{T}$ is self-adjoint. Thus (3) means that $\overline{A \upharpoonright \mathscr{D}} = A$ and $\overline{B \upharpoonright \mathscr{D}} = B$.

It is easy to check that $d(A, B)$ does not depend on the choice of $\alpha \in \mathbf{R}_1 \setminus \sigma(A)$, $\beta \in \mathbf{R}_1 \setminus \sigma(B)$. Moreover, A and B commute strongly if and only if X and Y commute, that is, $d(A, B)=0$.

Lemma 3. *$\mathscr{D}(A, B)$ is the largest linear subspace of $\mathscr{H}$ satisfying* (1). *Moreover, $\mathscr{D}(A, B) \equiv XY(I-P)\mathscr{H} = YX(I-P)\mathscr{H} \equiv \mathscr{D}(B, A)$.*

Proof. Suppose that $\varphi \in \mathscr{D}(AB) \cap \mathscr{D}(BA)$ and $AB\varphi = BA\varphi$. Then, $\varphi = XY\xi = YX\eta$ for some $\xi, \eta \in \mathscr{H}$. $(A-\alpha)(B-\beta)\varphi = \eta$ and $(B-\beta)(A-\alpha)\varphi = \xi$ imply that $\xi = \eta$. Hence $0 = \langle (XY-YX)\xi, \psi \rangle = \langle \xi, -(XY-YX)\psi \rangle$ for all $\psi \in \mathscr{H}$, i.e. $\xi \perp P\mathscr{H}$ and thus $\varphi = XY(I-P)\xi \in \mathscr{D}(A, B)$.

Conversely, let $\varphi = XY(I-P)\xi$ for some $\xi \in \mathscr{H}$. In particular, $\langle (I-P)\xi, -(XY-YX)^2(I-P)\xi \rangle = 0 = \|(XY-YX)(I-P)\xi\|^2$. Therefore, $\varphi = XY(I-P)\xi = YX(I-P)\xi$ which gives $AB\varphi = BA\varphi$. Moreover, this shows that $XY(I-P)\mathscr{H} \subseteq YX(I-P)\mathscr{H}$. Replacing XY by YX, we get $YX(I-P)\mathscr{H} \subseteq XY(I-P)\mathscr{H}$ thus completing the proof.

Lemma 4. *$\mathscr{D}(A, B)$ is dense in $\mathscr{H}$ if and only if $\overline{\mathscr{R}([X, Y])} \cap \mathscr{R}(XY) = \{0\}$.*

Proof. $\mathscr{D}(A, B) \equiv \mathscr{R}(YX(I-P))$ is dense if and only if $\mathscr{N}((YX(I-P))^*) = \mathscr{N}((I-P)XY) = \{0\}$. Obviously, $\varphi \in \mathscr{N}((I-P)XY)$ is equivalent to $XY\varphi \in \overline{\mathscr{R}([X, Y])}$. This gives the assertion, because $\mathscr{N}(X) = \mathscr{N}(Y) = \{0\}$.

Lemma 5. *$A \restriction \mathscr{D}(A, B)$ is e.s.a. if and only if*

$$P\mathscr{H} \cap \mathscr{D}(B) = \overline{\mathscr{R}([X, Y])} \cap \mathscr{R}(Y) = \{0\}.$$

$B \restriction \mathscr{D}(A, B)$ is e.s.a. if and only if

$$P\mathscr{H} \cap \mathscr{D}(A) = \overline{\mathscr{R}([X, Y])} \cap \mathscr{R}(X) = \{0\}.$$

Proof. We only prove the first assertion. Since $(A-\alpha)^{-1} = X$ is a bounded self-adjoint operator, $A \restriction \mathscr{D}(A, B)$ is e.s.a. if and only if $(A-\alpha)\mathscr{D}(A, B) \equiv Y(I-P)\mathscr{H} \equiv \mathscr{R}(Y(I-P))$ is dense in $\mathscr{H}$ or equivalently if $\mathscr{N}((I-P)Y) = \{0\}$. Since $\mathscr{N}(Y) = \{0\}$, the latter is equivalent to $P\mathscr{H} \cap \mathscr{R}(Y) = \{0\}$, which completes the proof.

In case that $\mathscr{R}([X, Y])$ is closed. the next Lemma gives a characterization of the class $\mathfrak{N}_1$ only in terms of domains.

Lemma 6. *If $\{A, B\} \in \mathfrak{N}_1$, then $\mathscr{D}(AB) \cap \mathscr{D}(A) = \mathscr{D}(BA) \cap \mathscr{D}(B) = \mathscr{D}(AB) \cap \mathscr{D}(BA) = \mathscr{D}(A, B)$ (and, by definition, this domain is dense in $\mathscr{H}$). Conversely, suppose that $\mathscr{R}([X, Y)]$ is closed. If $\mathscr{D}(A, B)$ is dense in $\mathscr{H}$ and $\mathscr{D}(AB) \cap \mathscr{D}(A) = \mathscr{D}(BA) \cap \mathscr{D}(B) = \mathscr{D}(AB) \cap \mathscr{D}(BA)$, then $\{A, B\} \in \mathfrak{N}_1$.*

Proof. Suppose that $\{A, B\}\in\mathfrak{N}_1$. Since $XY(I-P)\mathscr{H}=YX(I-P)\mathscr{H}$ by Lemma 3, it is clear that $\mathscr{D}(AB)\cap\mathscr{D}(A)\supseteqq\mathscr{D}(AB)\cap\mathscr{D}(BA)=YX\mathscr{H}\cap XY\mathscr{H}\supseteqq$ $\supseteqq XY(I-P)\mathscr{H}=\mathscr{D}(A, B)$. Now let $\varphi=YX\xi=X\eta\in\mathscr{D}(AB)\cap\mathscr{D}(A)$. Then, $X(\eta-Y\xi)=[X, Y](-\xi)$. Since $\{A, B\}\in\mathfrak{N}_1$, $\mathscr{R}(X)\cap\overline{\mathscr{R}([X, Y])}=\{0\}$ by Lemma 5. Hence $X(\eta-Y\xi)=0$ and, since $\mathscr{N}(X)=\{0\}$, $\eta=Y\xi$. As in the proof of Lemma 3, $XY\xi=YX\xi$ implies that $\xi\perp\overline{\mathscr{R}([X, Y])}$. Therefore, $\xi=(I-P)\xi$ and $\varphi=$ $=YX(I-P)\xi\in\mathscr{D}(A, B)$ which proves that $\mathscr{D}(A, B)\supseteqq\mathscr{D}(AB)\cap\mathscr{D}(A)$. Changing the role of A and B, we get $\mathscr{D}(BA)\cap\mathscr{D}(B)=\mathscr{D}(A, B)$.

We now prove the second assertion. Set $\mathscr{D}=\mathscr{D}(A, B)$ in Definition 1. Then (1) and (2) are satisfied by Lemma 3, resp., by assumption. We show that $\mathscr{R}([X, Y])\cap\mathscr{R}(X)=\{0\}$. Suppose that $\varphi:=[X, Y]\xi=X\eta$ for some $\xi, \eta\in\mathscr{H}$. Then $\psi:=XY\xi-X\eta=YX\xi\in\mathscr{D}(A)\cap\mathscr{D}(AB)$. By assumption, $\psi\in\mathscr{D}(BA)$, that is, $\psi=XY\zeta$ for some $\zeta\in\mathscr{H}$. Hence $X(Y\xi-\eta-Y\zeta)=0$ which gives $\eta=Y(\xi-\zeta)$. Therefore, $\eta\in\mathscr{R}(Y)$ and $\varphi\in\mathscr{R}([X, Y])\cap\mathscr{R}(XY)$. Since we assumed that $\mathscr{D}(A, B)$ is dense, Lemma 4 gives $\varphi=0$. This proves $\mathscr{R}([X, Y])\cap\mathscr{R}(X)=\{0\}$. From Lemma 5 (recall that $\mathscr{R}([X, Y])$ is closed!) we conclude that $B\restriction\mathscr{D}(A, B)$ is e.s.a. Similarly, $A\restriction\mathscr{D}(A, B)$ is e.s.a. Thus $\{A, B\}\in\mathfrak{N}_1$.

Remarks. 1. If we do not assume that $\mathscr{D}(A, B)$ is dense, then the equality of the domains in Lemma 6 does not ensure that $\{A, B\}\in\mathfrak{N}_1$ in general. For an example, recall that there are unbounded self-adjoint operators A and B so that $\mathscr{D}(A)\cap\mathscr{D}(B)=\{0\}$ ([17]). Then, $\mathscr{D}(AB)\cap\mathscr{D}(A)=\mathscr{D}(BA)\cap\mathscr{D}(B)=\mathscr{D}(AB)\cap\mathscr{D}(BA)=$ $=\mathscr{D}(A, B)=\{0\}$, but $\{A, B\}\notin\mathfrak{N}_1$.

2. The second assertion in Lemma 6 is no longer true if we replace the density of $\mathscr{D}(A, B)$ by that of $\mathscr{D}(AB)\cap\mathscr{D}(BA)$. To prove this remark, let $\mathscr{H}=L^2(1, 2)$, let X be the multiplication operator by x, and let $Y=I+H/2$, where H is the finite Hilbert transform in $L^2(1, 2)$. Then X and Y have bounded inverses denoted by A resp. B. Moreover, $[X, Y]$ has rank one and $\mathscr{D}(AB)=\mathscr{D}(A)=\mathscr{D}(BA)=$ $=\mathscr{D}(B)=\mathscr{H}$, but $\{A, B\}\notin\mathfrak{N}_1$ by the first remark in 1.1.

3. It follows from the preceding that if $\{A, B\}\in\mathfrak{N}_1$, then we can take $\mathscr{D}=\mathscr{D}(A, B)$ in Definition 1.

1.3. Theorem 7. *Suppose* $\{A, B\}\in\mathfrak{N}_1$ *Suppose also that* $\alpha\in\mathbf{R}_1\setminus\sigma(A)$ *and* $\beta\in\mathbf{R}_1\setminus\sigma(B)$. *If* $X:=(A-\alpha)^{-1}$ *and* $Y:=(B-\beta)^{-1}$, *then*

$$(4)\quad \mathscr{N}(X)=\mathscr{N}(Y)=\{0\}\quad \text{and}\quad \overline{\mathscr{R}([X, Y])}\cap\mathscr{R}(X)=\overline{\mathscr{R}([X, Y])}\cap\mathscr{R}(Y)=\{0\}.$$

Conversely, if X *and* Y *are bounded self-adjoint operators satisfying* (4), *then* $\{A:=X^{-1}+\alpha, B:=Y^{-1}+\beta\}\in\mathfrak{N}_1$ *for* $\alpha, \beta\in\mathbf{R}_1$.

The proof of Theorem 7 follows immediately from the three Lemmas 3, 4 and 5 above.

2. Pairs with defect number one

In this section we classify, up to unitary equivalence, all irreducible pairs $\{A, B\} \in \mathfrak{N}_1$ with defect number one for which $\sigma(A) \neq \mathbf{R}_1$ and $\sigma(B) \neq \mathbf{R}_1$.

2.1. We first collect some facts concerning bounded operators with rank one self-commutators. A very readable account of this theory is given in [3]; see also [11], [2], [8], [4].

A bounded operator T in $\mathscr{H}$ is said to be *completely hyponormal* if T is hyponormal (that is, $[T^*, T] \geqq 0$) and if T has no nontrivial reducing subspace on which it is normal. Suppose that T is completely hyponormal and that $\dim \mathscr{R}([T^*, T]) = 1$. The essential underlying result for our study is the following. There is a function $g(x, y) \in L^1(\mathbf{R}_2)$ with compact support such that

$$\operatorname{Tr} i[p(X, Y), q(X, Y)] = \frac{1}{2\pi} \iint \left(\frac{\partial p}{\partial x} \frac{\partial q}{\partial y} - \frac{\partial p}{\partial y} \frac{\partial q}{\partial x} \right) g(x, y)\, dx\, dy \tag{5}$$

for all polynomials p and q in X and Y. Moreover, $0 \leqq g(x, y) \leqq 1$ a.e. on $\mathbf{R}_2$. $g = g_T$ is called the *principal function of* T. It was introduced by Pincus [11]. g_T is a complete unitary invariant for T, that is, two completely hyponormal operators T and $\tilde{T}$ with rank one self-commutators are unitarily equivalent if and only if their principal functions g_T and $g_{\tilde{T}}$ coincide (considered as elements of $L^1(\mathbf{R}_2)$). Moreover, for each function $g \in L^1(\mathbf{R}_2)$ with compact support satisfying $0 \leqq g \leqq 1$ there exists a completely hyponormal operator T with principal function g such that $\dim \mathscr{R}([X, Y]) = 1$.

We now return to the class $\mathfrak{N}_1$. Let $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ be pairs of the class $\mathfrak{N}_1$ in Hilbert spaces $\mathscr{H}$ and $\tilde{\mathscr{H}}$. $\{A, B\}$ is called *unitarily equivalent* to $\{\tilde{A}, \tilde{B}\}$ if there is an isometry U of $\mathscr{H}$ onto $\tilde{\mathscr{H}}$ such that $A = U^* \tilde{A} U$ and $B = U^* \tilde{B} U$. In that case we clearly have $U\mathscr{D}(A, B) = \mathscr{D}(\tilde{A}, \tilde{B})$. As in Section 1, we assume that $\alpha \in \mathbf{R}_1 \setminus \sigma(A)$ and $\beta \in \mathbf{R}_1 \setminus \sigma(B)$. It is easy to see that $\{A, B\}$ is unitarily equivalent to $\{\tilde{A}, \tilde{B}\}$ if and only if $\alpha \notin \sigma(\tilde{A})$, $\beta \notin \sigma(\tilde{B})$ and $T := X + iY = (A - \alpha)^{-1} + i(B - \beta)^{-1}$ is unitarily equivalent to $\tilde{T} := \tilde{X} + i\tilde{Y} = (\tilde{A} - \alpha)^{-1} + i(\tilde{B} - \beta)^{-1}$. Moreover, $\{A, B\}$ is irreducible if and only if T is irreducible.

Suppose now in addition that $d(A, B) = 1$. Then the self-adjoint operator $D := [T^*, T]$ has rank one and therefore either $D \geqq 0$ or $D \leqq 0$. Obviously, $\operatorname{sign} D$ is a unitary invariant for $\{A, B\} \in \mathfrak{N}_1$. By changing the role of A and B in case $D \leqq 0$, we can restrict ourselves to pairs $\{A, B\} \in \mathfrak{N}_1$ for which $D = [T^*, T] \geqq 0$, that is, T is hyponormal. Since D has rank one, $\{A, B\} \in \mathfrak{N}_1$ is irreducible (or equivalently, T is irreducible) if and only if T is completely hyponormal. Therefore, under the above assumptions (i.e., $d(A, B) = 1$, $\alpha \in \mathbf{R}_1 \setminus \sigma(A)$, $\beta \in \mathbf{R}_1 \setminus \sigma(B)$ and $[T^*, T] \geqq 0$), g_T is a complete unitary invariant for irreducible $\{A, B\} \in \mathfrak{N}_1$.

To proceed in the converse direction, it still remains to decide when a given completely hyponormal operator with rank one self-commutator leads to an (irreducible) pair $\{A, B\}$ in $\mathfrak{N}_1$. The answer is contained in

Theorem 1. *Suppose that T is a completely hyponormal operator in the Hilbert space $\mathscr{H}$ with rank one self-commutator. Let g be the principal function of $T, X=\operatorname{Re} T$ and $Y=\operatorname{Im} T$. Let $x_0, y_0 \in \mathbf{R}_1$. Then the inverses $A:=(X-x_0)^{-1}$ and $B:=(Y-y_0)^{-1}$ exist. $\{A, B\} \in \mathfrak{N}_1$ if and only if $\iint g(x, y)(x-x_0)^{-2}\, dx\, dy = \iint g(x, y)(y-y_0)^{-2} dx\, dy = +\infty$.*

2.2. In the proof of Theorem we use the following easy

Lemma 2. *Let N be a bounded operator in $\mathscr{H}$ and let $\varphi, \psi \in \mathscr{H}$. Let $\{z_n, n \in \mathbf{N}\}$ be a zero sequence of complex numbers such that $z_n \notin \sigma(N)$ for all $n \in \mathbf{N}$.*

(i) *If $\lim_n (N-z_n)^{-1}\psi = \varphi$, then $N\varphi = \psi$.*

(ii) *Suppose that N is normal and that $\mathscr{N}(N)=\{0\}$. Suppose that*

$$c := \sup_{n \in \mathbf{N}} |z_n| (\operatorname{dist}(z_n, \sigma(N)))^{-1} < \infty.$$

If $N\varphi = \psi$, then $\lim_n (N-z_n)^{-1}\psi = \varphi$.

Proof. (i) $\psi = N(N-z_n)^{-1}\psi - z_n(N-z_n)^{-1}\psi \to N\varphi - 0\varphi$ as $n \to \infty$, i.e., $N\varphi = \psi$.

(ii) Letting $N = \int z\, dG(z)$ be the spectral decomposition for N, we have

$$\|(I-G(\{0\}))z_n(N-z_n)^{-1}\varphi\|^2 = \int_{\sigma(N) \setminus \{0\}} |z_n|^2 |z-z_n|^{-2} d\|G(z)\varphi\|^2 \leqq \int c^2 d\|G(z)\varphi\|^2 < \infty.$$

The dominated Lebesgue theorem yields $\lim_n (I-G(\{0\}))z_n(N-z_n)^{-1}\varphi = 0$. Since $\mathscr{N}(N)=\{0\}$, we have $G(\{0\})=0$ and therefore $\lim_n z_n(N-z_n)^{-1}\varphi = 0$. Therefore, $(N-z_n)^{-1}\psi = z_n(N-z_n)^{-1}\varphi + \varphi \to 0 + \varphi$ as $n \to \infty$, which completes the proof.

Proof of Theorem 1. Since T is hyponormal and $\dim \mathscr{R}([T^*, T])=1$, there is a vector $\xi \neq 0$ in $\mathscr{H}$ such that $[T^*, T] = 2i[X, Y] = \xi \otimes \xi$. Since T is completely hyponormal, the operators X and Y are absolutely continuous ([12], Theorem 2.2.4 or [3], Theorem 3.2). In particular, $\mathscr{N}(X-x_0) = \mathscr{N}(Y-y_0) = \{0\}$. Hence the inverses A and B exist and are self-adjoint operators in $\mathscr{H}$.

By induction it follows that $2i[X^{n+1}, Y] = \sum_{j=0}^{n} X^j \xi \otimes X^{n-j}\xi$ for $n \in N_0$. Therefore, $\operatorname{Tr} 2i[X^{n+1}, Y] = (n+1)\langle X^n \xi, \xi\rangle$. Applying the tracial functional calculus in case $p(X, Y) = X^{n+1}$, $q(X, Y) = Y$, we get

$$\operatorname{Tr} i[X^{n+1}, Y] = (1/2\pi) \iint (n+1) x^n g(x, y)\, dx\, dy.$$

Putting both together, we conclude that

$$\pi\langle p(X)\xi, \xi\rangle = \iint p(x)g(x, y)\,dx\,dy \tag{6}$$

for each complex polynomial $p(\cdot)$ in X.

Setting $N:=X-x_0$ and $z_n:=2^{-n}i$ for $n\in\mathbf{N}$, the assumptions of Lemma 2 are fulfilled. Therefore, by Lemma 2, $\xi\in\mathscr{R}(X-x_0)$ if and only if the sequence $\varphi_n:=(X-x_0+z_n)^{-1}\xi$, $n\in\mathbf{N}$, converges in $\mathscr{H}$. We prove that

$$\xi\in\mathscr{R}(X-x_0) \quad \text{if and only if} \quad \iint g(x, y)(x-x_0)^{-2}\,dx\,dy<\infty.$$

For simplicity in notation, let $x_0=0$. Assume that $\{\varphi_n\}$ converges. Take a positive number L so that $\operatorname{supp} g\subseteqq[-L, L]\times\mathbf{R}_1$ and $\sigma(X)\subseteqq[-L, L]$. Let $n\in\mathbf{N}$. If we approximate the continuous function $|x+2^{-n}i|^{-2}$ by polynomials uniformly on $[-L, L]$, it follows from (6) that

$$\pi\|\varphi_n\|^2 = \pi\|(X+2^{-n}i)^{-1}\xi\|^2 = \iint g(x, y)\,|x+2^{-n}i|^{-2}\,dx\,dy.$$

By Fatou's lemma (recall that $g(x, y)\geqq 0$ a.e.), we obtain

$$\iint g(x, y)x^{-2}dx\,dy \leqq \lim_n \iint g(x, y)|x+2^{-n}i|^{-2}\,dx\,dy = \pi \lim_n \|\varphi_n\|^2 <\infty.$$

Conversely, assume that $\iint g(x, y)x^{-2}dx\,dy<\infty$. Again uniform approximation by polynomials yields

$$\pi\,\|\varphi_n-\varphi_m\|^2 = \iint g(x, y)\,|(x+2^{-n}i)^{-1}-(x+2^{-m}i)^{-1}|^2\,dx\,dy \tag{7}$$

for $n, m\in\mathbf{N}$. Since $|x+2^{-n}i|^{-1}\leqq|x|^{-1}$ for $n\in\mathbf{N}$ and the above integral is finite, Lebesgue's dominated convergence theorem (in the formulation given in [1], IV, § 3, 7.) applies and gives

$$\lim_n \iint g(x, y)\,|(x+2^{-n}i)^{-1}-x^{-1}|^2\,dx\,dy = 0.$$

Because of (7), the latter implies that $\{\varphi_n\}$ is a Cauchy sequence in $\mathscr{H}$. Therefore, $\{\varphi_n\}$ converges in $\mathscr{H}$. This completes the proof of (5).

Similarly, $\xi\in\mathscr{R}(Y-y_0)$ if and only if $\iint g(x, y)(y-y_0)^{-2}dx\,dy<\infty$. Theorem 1.7 now gives the result.

2.3. Remarks. 1. Each hyponormal operator on a separable Hilbert space can be represented as a singular integral operator on a direct integral Hilbert space ([16], [11]; see [3], 2.3). This result can be used to obtain concrete realizations for pairs $\{A, B\}\in\mathfrak{N}_1$ with $d(A, B)=1$, $\sigma(A)\neq\mathbf{R}_1$ and $\sigma(B)\neq\mathbf{R}_1$.

2. Given a function $g\in L^1(\mathbf{R}_2)$ with compact support satisfying $0\leqq g\leqq 1$ and $\iint gx^{-2}dx\,dy=\iint gy^{-2}dx\,dy=\infty$, there exists an (unique up to unitary equivalence) irreducible pair $\{A:=X^{-1}, B:=Y^{-1}\}\in\mathfrak{N}_1$ with defect number one such that g is the principal function of the completely hyponormal operator $T=X+iY$.

This follows from Theorem 1 and the discussion before Theorem 1. Especially, we see that there is a large variety of irreducible pairs of the class $\mathfrak{N}_1$ even in the case $d(A, B)=1$.

3. We illustrate Theorem 1 by a well-known general example (see, for example, [3]). Let $\mathcal{J}$ be a Lebesgue measurable bounded subset of $\mathbf{R}_1$. Let $a, b \in L^\infty(\mathcal{J})$, where $b(t)\neq 0$ a.e. on $\mathcal{J}$ and $a(t)$ is real valued. Define an operator T on $\mathcal{H}=L^2(\mathcal{J})$ by

$$(Tf)(t) \equiv ((X+iY)f)(t) := tf(t)+i\left[a(t)f(t)+\frac{b(t)}{\pi i}\int_{\mathcal{J}}\frac{b(s)f(s)}{t-s}\,ds\right].$$

Then, $[T^*, T]=(2/\pi)b\otimes b$. By a result of XA DAO-XENG [16], T is the most general completely hyponormal operator with rank one self-commutator whose real part has a cyclic vector. The principal function of T is the characteristic function of the set

$$\mathcal{E}=\{(x, y)\in \mathbf{R}_2\colon\ x\in\mathcal{J}\ \text{ and }\ a(x)-|b(x)|^2\leqq y\leqq a(x)+|b(x)|^2\}.$$

If $(x_0, y_0)\in\mathbf{R}_2$ is in the interior of $\mathcal{E}$, then the conditions in Theorem 1 are, of course, satisfied and thus $\{A:=(X-x_0)^{-1}, B:=(Y-y_0)^{-1}\}\in\mathfrak{N}_1$. In that case $\sigma(A)\supseteq(-\infty, -L)\cup(L, +\infty)$ and $\sigma(B)\supseteq(-\infty, L)\cup(L, +\infty)$ for some $L>0$. If (x_0, y_0) is not in the closure of $\mathcal{E}$, then $\{A, B\}\notin\mathfrak{N}_1$.

We now discuss two rather simple (but typical) examples for which (x_0, y_0) is in the boundary of $\mathcal{E}$. First suppose x_0 is in the interior of $\mathcal{J}$ and $a(x_0)-|b(x_0)|^2= =y_0<a(x_0)+|b(x_0)|^2$. Assume that $a+|b|^2$ is continuous at x_0. Assume that $a(x)-|b|(x)|^2=\lambda|x|^\alpha$ for $x_0-\varepsilon<x<x_0$ and that $a(x)-|b(x)|^2=\mu|x|^\beta$ for $x_0<x< <x_0+\varepsilon$ with $\lambda, \mu\in\mathbf{R}_1, \alpha>0, \beta>0$ and $\varepsilon>0$. Then, $\{A, B\}\in\mathfrak{N}_1$ if and only if $\lambda>0, \mu>0, \alpha<1$ and $\beta<1$. Moreover, $\sigma(A)\supseteq(-\infty, -L)\cup(L, +\infty)$ for some $L>0$.

In the second example we assume that $\mathcal{J}\subseteq(x_0, +\infty)$, $(x_0, x_0+\varepsilon)\subseteq\mathcal{J}$ for some $\varepsilon>0$ and $a(x)-|b(x)|^2\geqq 0$ on $\mathcal{J}$. Suppose that $a(x)-|b(x)|^2=\lambda|x|^\alpha$ and $a(x)+|b(x)|^2=\mu|x|^\beta$ for $x\in(x_0, x_0+\varepsilon)$ with $\lambda>0, \mu>0, \alpha>0$ and $\beta>0$. Then $\{A, B\}\in\mathfrak{N}_1$ if and only if $\beta\leqq 1\leqq\alpha$. (Note that $\lambda<\mu$ in case $\alpha=\beta$, since $b(x)\neq 0$ a.e. on $\mathcal{J}$.) In this example A and B are positive operators.

3. Toeplitz operators

3.1. Let $L^2=L^2(\mathbf{T})$ be the L^2-space on the unit circle $\mathbf{T}$ with normalized Lebesgue measure. Let H^2 be the usual Hardy space on $\mathbf{T}$ and let P_+ denote the orthogonal projection on H^2. If $\varphi\in L^\infty(\mathbf{T})$, then the Toeplitz operator T_φ is defined by $T_\varphi f=P_+\varphi f$ for $f\in H^2$. Let $\varphi=\sum_{n=-\infty}^{\infty}\varphi_n e_n$ be the Fourier expansion of φ, where $e_n:=e^{int}$ for $n\in\mathbf{Z}$. The matrix $(d_{nk})_{n,k\in\mathbf{N}_0}$ of the self-commutator $D=[T_\varphi^*, T_\varphi]$ with respect to the orthonormal base $(e_n)_{n\in\mathbf{N}_0}$ of H^2 is given by

$$d_{nk} = \langle De_k, e_n\rangle = \sum_{j=0}^{\infty} \varphi_{j-k}\overline{\varphi_{j-n}} - \varphi_{n-j}\overline{\varphi_{k-j}}. \tag{8}$$

As usual, we identify H^2 and $l_2(\mathbf{N}_0)$ via Fourier expansion. If $\varphi\in H^\infty$, then $\varphi_n=0$ for all $n<0$ and (8) yields

$$d_{nk} = \langle (S^*)^{n+1}\varphi, (S^*)^{k+1}\varphi\rangle \quad \text{for all } k, n\in\mathbf{N}_0. \tag{9}$$

Here S^* is the backward shift in $l_2(\mathbf{N}_0)$ which is defined by $S^*(\psi_0, \psi_1, \psi_2, \ldots)=$ $=(\psi_1, \psi_2, \ldots)$.

Theorem 1. *Let $p(z)=\sum a_j z^j$ be a non-constant complex polynomial of degree n. Let $X=\operatorname{Re} T_p$, $Y=\operatorname{Im} T_p$, $A=X^{-1}$ and $B=Y^{-1}$. Then $\{A, B\}\in\mathfrak{N}_1$ if and only if each $z\in\mathbf{C}_1$, $z\neq 0$, satisfying*

$$(\bar{p}(z)+p(z))(\bar{p}(z)-p(z)) = 0 \tag{10}$$

lies on $\mathbf{T}$, where $\bar{p}(z):=\sum \bar{a}_j z^{-j}$. If this is true, then $d(A, B)=n$.

Proof. Let $\mathscr{K}_n:=\operatorname{Lin}\{e_j: j=0, \ldots, n-1\}$. First we check that $\mathscr{R}([X, Y])\equiv$ $\equiv\mathscr{R}([T_p^*, T_p])\equiv\mathscr{R}(D)=\mathscr{K}_n$. From (9) it is obvious that $DH^2\subseteq\mathscr{K}_n$. Since $a_n\neq 0$, it follows that the n vectors $(S^*)^n p=(a_n, 0, \ldots)$, $(S^*)^{n-1}p=(a_{n-1}, a_n, 0, \ldots)$, … …, $S^*p=(a_1, \ldots, a_n, 0, \ldots)$ are linearly independent. That is, the Gram determinant $\det(d_{nk})_{n,k=0,\ldots,n-1}$ is non-zero. Therefore, the map $D: \mathscr{K}_n\to\mathscr{K}_n$ is one-to-one and $\mathscr{R}([X, Y])\equiv\mathscr{R}(D)=\mathscr{K}_n$.

Clearly, $2X=T_{\bar{p}+p}$ and $2Y=T_{i(\bar{p}-p)}$. It is well-known that each nonzero bounded self-adjoint Toeplitz operator has trivial kernel ([7], Ex. 198). Hence $\mathscr{N}(X)=\mathscr{N}(Y)=\{0\}$ and the operators $A:=X^{-1}$ and $B:=Y^{-1}$ exist.

Set $q(z)=z^n(\bar{p}(z)+p(z))$. Let $f\in H^2$. Obviously, $Xf\in\mathscr{K}_n$ is equivalent to $0=\langle Xf, e_{n+k}\rangle$ for all $k\in\mathbf{N}_0$. Since $\bar{p}+p$ is real on $\mathbf{T}$, this is equivalent to $0=\langle T_{\bar{p}+p}f, e_{n+k}\rangle=\langle P_+(\bar{p}+p)f, e_{n+k}\rangle=\langle f, (\bar{p}+p)e_{n+k}\rangle=\langle f, S^k(\bar{p}+p)e_n\rangle=\langle f, S^k q\rangle$ for $k\in\mathbf{N}_0$. Combined with $\mathscr{N}(X)=\{0\}$, this shows that $\mathscr{R}(X)\cap\mathscr{K}_n=\{0\}$ if and only if $q(z)$ is a cyclic vector for the shift operator, that is, if $q(z)$ is an outer function in H^2. But a polynomial is an outer function in H^2 if and only if it does not vanish in the set $\{z\in\mathbf{C}_1: |z|<1\}$. Since $\mathscr{R}([X, Y])=\mathscr{K}_n$ as we have seen above, it follows that $\mathscr{R}(X)\cap\overline{\mathscr{R}([X, Y])}=\mathscr{R}(X)\cap\mathscr{K}_n=\{0\}$ if and only if $q(z)$ has no zeros in $\{z\in\mathbf{C}_1: |z|<1\}$. Obviously, $q(z)=0$ for a complex number $z\neq 0$ implies that $q(\bar{z}^{-1})=0$. Moreovei, $q(0)\neq 0$, since $a_n\neq 0$. Therefore, $\mathscr{R}(X)\cap\overline{\mathscr{R}([X, Y])}=\{0\}$ if and only if each solution $z\in\mathbf{C}_1$, $z\neq 0$, of $\bar{p}(z)+p(z)=0$ lies on the unit circle. Similarly, $\mathscr{R}(Y)\cap\overline{\mathscr{R}([X, Y])}=\{0\}$ if and only if $\bar{p}(z)-p(z)=0$ for some $z\in\mathbf{C}_1$, $z\neq 0$, implies $|z|=1$.

Now the assertion follows immediately from Theorem 1.7.

Corollary 2. *If all zeros of* $p(z)$ *are in* $\{z\in\mathbf{C}_1\colon |z|\leqq 1\}$, *then* $\{A, B\}\in\mathfrak{N}_1$.

Proof. Let $b_1, \ldots, b_n$ be the zeros of $p(z)$. Assume the contrary, that is, $\{A, B\}\notin\mathfrak{N}_1$. Then, by Theorem 1, there is a non-zero $z\in\mathbf{C}_1$, $|z|\neq 1$, satisfying (10). Hence $|p(z)|=|\bar{p}(z)|$. Since $|p(\bar{z}^{-1})|=|\bar{p}(\bar{z}^{-1})|$, we can assume that $|z|<1$. From

$$\left|a_n \prod_{j=1}^{n}(z-b_j)\right| = |p(z)| = |\bar{p}(z)| = \left|\overline{a_n} \prod_{j=1}^{n}\left(\frac{1}{z}-\overline{b_j}\right)\right|$$

we obtain $|z|^n \prod_{j=1}^{n} |(b_j-z)(1-b_jz)^{-1}|=1$. Since $|b_j|\leqq 1$ by assumption and $|z|<1$, it follows that $|(b_j-z)(1-b_jz)^{-1}|\leqq 1$ and $|z|^n<1$, and we have our contradiction.

Examples. 1. Applying Corollary in case $p(z)\equiv z$, we see that the shift operator $S=T_z$ gives a pair $\{A=(\mathrm{Re}\, S)^{-1}, B=(\mathrm{Im}\, S)^{-1}\}\in\mathfrak{N}_1$ with defect number one. This could be verified directly or obtained from Section 2 as well.

2. Let $p_1(z)=2(z-\sqrt{2})$ and let $p_2(z)=(1+i)p_1(z)/2=(1+i)(z-\sqrt{2})$. Let $X_j=\mathrm{Re}\, T_{p_j}$ and $Y_j=\mathrm{Im}\, T_{p_j}$ for $j=1,2$. Then $X_1=X_2+Y_2$ and $Y_1=Y_2-X_2$. In case $p_2(z)$ the solutions of (10) are given by $z_1=z_2=(1+i)/\sqrt{2}$, $z_3=z_4=(1-i)/\sqrt{2}$. Since they are all of modulus one, $\{X_2^{-1}, Y_2^{-1}\}\in\mathfrak{N}_1$ by Theorem 1. Since the zero of $p_1(z)$ is not contained in $\{|z|\leqq 1\}$, we see that the condition given in Corollary 2 is sufficient, but not necessary. On the other hand, $z_0=\sqrt{2}+1$ is a solution of (10) for $p_1(z)$. Consequently, $\{X_1^{-1}, Y_1^{-1}\}\equiv\{(X_2+Y_2)^{-1}, (Y_2-X_2)^{-1}\}$ does not belong to the class $\mathfrak{N}_1$. This also follows from the fact that X_1^{-1} is bounded, because $\sigma(X_2)=[-2-2\sqrt{2}, 2-2\sqrt{2}]$.

3.2. A further study of the function $\varphi\in L^\infty$ for which $X=\mathrm{Re}\, T_\varphi$ and $Y=\mathrm{Im}\, T_\varphi$ satisfy the conditions of Theorem 1.7 seems to be of some interest. As a result in the opposite direction we mention

Proposition 3. *Suppose that* $\varphi\in H^\infty$ *is a cyclic vector for the backward shift operator. If the sequence* $(\|(S^*)^n\varphi\|)_{n\in\mathbf{N}}$ *is in* $l_2(\mathbf{N})$, *then* $\mathscr{R}([T_\varphi^*, T_\varphi])$ *is dense in* H^2 *(and the conditions* (4) *in Theorem* 1.7 *are certainly not fulfilled).*

Proof. Since $D=[T_\varphi^*, T_\varphi]$ is self-adjoint, it suffices to show that D has trivial kernel. Suppose that $Df=[T_\varphi^*, T_\varphi]f=0$ for some function $f=\sum_{j=0}^{\infty} f_je_j\in H^2$. From $(\|(S^*)^n\varphi\|)\in l_2$ and $(f_n)\in l_2$ it follows that the series $\sum_{k=0}^{\infty} f_k(S^*)^{k+1}\varphi$ converges in the Hilbert space H^2 and thus defines a function $h\in H^2$.

$Df=0$ implies that $\sum_{k=0}^{\infty} d_{nk}f_k=\sum_{k=0}^{\infty}\langle (S^*)^{n+1}\varphi, (S^*)^{k+1}\varphi\rangle f_k=\langle (S^*)^{n+1}\varphi, h\rangle=$ $=\langle (S^*)^n\varphi, Sh\rangle=0$ for all $n\in\mathbf{N}_0$. Because φ is cyclic for S^*, this yields $Sh=0$

and hence $h=0$. This gives $0=\langle e_n, h\rangle=\sum_{k=0}^{\infty} f_k\overline{\varphi_{k+n+1}}=\langle Sf, (S^*)^n_\varphi\rangle$ for $n\in\mathbf{N}_0$. Again by the cyclicity of φ for S^* it follows that $Sf=0$ and $f=0$ thus completing the proof.

Remark. Examples of functions φ as in Proposition 3 are easily obtained by taking lacunary series. Let $(n_k)_{k\in\mathbf{N}}$ be a sequence of positive integers such that $n_{k+1}\geqq\lambda n_k$ for some $\lambda>1$. If $(a_k)_{k\in\mathbf{N}}$ is an l_2-sequence such that $a_k\neq 0$ for all $k\in\mathbf{N}$, then $\varphi:=\sum_{k=1}^{\infty} a_k e_{n_k}\in H^2$ is cyclic for S^* ([5]), Theorem 2.5.5). The assumption $(\|(S^*)^n\varphi\|)\in l_2$ can be fulfilled by choosing a_k sufficiently small for large k.

4. Perturbations of normal operators

4.1. Suppose that $\{A, B\}\in\mathfrak{N}_1$. As in Section 1, we assume that $\alpha\in\mathbf{R}_1\setminus\sigma(A)$ and $\beta\in\mathbf{R}_1\setminus\sigma(B)$ and we set $X=(A-\alpha)^{-1}$ and $Y=(B-\beta)^{-1}$. It is easy to check that A and B commute strongly if and only if the bounded operator $T=X+iY$ is normal. Another method to construct couples of the class $\mathfrak{N}_1$ (with non-zero defect numbers!) is the following. We "perturb" the normality of T by adding an appropriate operator, say R, and we then take the inverses of the real and imaginary parts of $T+R$. We will discuss this method in the case $R=-NE$, where E is a rank one projection.

We denote by $U_r(x, y)$ the closed disk of radius r centered at $(x, y)\in\mathbf{R}_2$, and by K_r the circle of radius r centered at the origin.

Theorem 1. *Let N be a bounded normal operator with spectral resolution $N=\int z\,dG(z)$. Let $X=\operatorname{Re} N$ and $Y=\operatorname{Im} N$. Suppose that $\mathcal{N}(X)=\mathcal{N}(Y)=\{0\}$. Let $\xi\in\mathcal{H}$, $\|\xi\|=1$. Let $\tilde{X}:=\operatorname{Re} N(I-E_\xi)$ and $\tilde{Y}:=\operatorname{Im} N(I-E_\xi)$, where $E_\xi:=\xi\otimes\xi$. If ξ can be chosen such that $\tilde{X}$ and $\tilde{Y}$ satisfy the condition* (4) *in Theorem* 1.7 *and if $d(\tilde{A}:=\tilde{X}^{-1}, \tilde{B}:=\tilde{Y}^{-1})\neq 0$, then either $d(\tilde{A}, \tilde{B})=2$ or $d(\tilde{A}, \tilde{B})=3$.*

(i) *There is a $\xi\in\mathcal{H}$ such that $d(\tilde{A}, \tilde{B})=3$ if and only if the (self-adjoint) operators $(X-a)(X-b)Y^{-1}$ and $(Y-a)(Y-b)X^{-1}$ are unbounded for all $a, b\in\mathbf{R}_1$.*

(ii) *There exists a $\xi\in\mathcal{H}$ such that $d(\tilde{A}, \tilde{B})=2$ if and only if there is an $r>0$ such that the (self-adjoint) operators $(X-a)Y^{-1}\restriction\mathcal{H}_r$ and $(Y-a)X^{-1}\restriction\mathcal{H}_r$ are unbounded for all $a\in\mathbf{R}_1$ (or equivalently, the points* $(0, r)$, $(0, -r)$, $(r, 0)$ *and* $(-r, 0)$ *are in the spectrum of $N\restriction\mathcal{H}_r$), where $\mathcal{H}_r:=G(K_r)\mathcal{H}$.*

4.2. Proof. We will denote by $\mathcal{K}$ the linear span of the vectors $\xi, N\xi$ and $N^*N\xi$ in $\mathcal{H}$.

Suppose that $\xi\in\mathcal{H}$, $\|\xi\|=1$, is chosen such that $\tilde{X}$ and $\tilde{Y}$ satisfy (4). Suppose that $d(\tilde{A}=\tilde{X}^{-1}, \tilde{B}=\tilde{Y}^{-1})\neq 0$. By the normality of N, we have $2i[\tilde{X}, \tilde{Y}]=$ $=[(N(I-E_\xi))^*, N(I-E_\xi)]=(I-E_\xi)N^*N(I-E_\xi)-N(I-E_\xi)N^*=(-N^*N\xi+\|N\xi\|^2\xi)\otimes$

$\otimes\xi-\xi\otimes N^*N\xi+N\xi\otimes N\xi$. From this formula we see that ξ cannot be an eigenvector of N, because in this case $[\tilde{X},\tilde{Y}]=0$ and thus $d(\tilde{A},\tilde{B})=0$. Moreover, $\mathscr{R}([\tilde{X},\tilde{Y}])\subseteqq\mathscr{K}$.

We begin with the case $d(\tilde{A},\tilde{B})=3$. Since $\mathscr{R}([\tilde{X},\tilde{Y}])\subseteqq\mathscr{K}$, we then have $\mathscr{R}([\tilde{X},\tilde{Y}])=\mathscr{K}$ and $\dim\mathscr{K}=3$. We show that $(Y-a)(Y-b)\xi\notin\mathscr{R}(X)$ for all $a,b\in\mathbf{R}_1$. Assume the contrary, that is, $(Y-a)(Y-b)\xi=X\eta$ for some $\eta\in\mathscr{H}$. Then, $N^*N\xi+(a+b)iN\xi+ab\xi=(Y-a)(Y-b)\xi+X^2\xi+(a+b)iX\xi=X\psi$ with

$$\psi:=\eta+(a+b)i\xi+X\xi$$

and

$$\tilde{X}\psi=X\psi-(1/2)\langle\psi,N\xi\rangle\xi-(1/2)\langle\psi,\xi\rangle N\xi=(\dots)\xi+(\dots)N\xi+N^*N\xi.$$

Since $\dim\mathscr{K}=3$, $\xi, N\xi$ and $N^*N\xi$ are linearly independent and hence $\tilde{X}\psi\neq 0$. On the other hand, $\tilde{X}\psi\in\mathscr{R}(\tilde{X})\cap\mathscr{K}=\mathscr{R}(\tilde{X})\cap\mathscr{R}([\tilde{X},\tilde{Y}])$. This contradicts (4). Therefore, $(Y-a)(Y-b)\xi\notin\mathscr{R}(X)$ which implies that $(Y-a)(Y-b)X^{-1}$ is unbounded for all $a,b\in\mathbf{R}_1$. The proof for $(X-a)(X-b)Y^{-1}$ is similar.

Now assume that $0<d(\tilde{A},\tilde{B})<3$. If the vectors $-N^*N\xi+\|N\xi\|^2\xi$, $-\xi$ and $N\xi$ would be linearly independent, then we could choose vectors $\varphi\in\mathscr{H}$ orthogonal to two of these vectors, but not orthogonal to the third. This would imply that $\dim\mathscr{R}([\tilde{X},\tilde{Y}])=d(\tilde{A},\tilde{B})=3$ which contradicts our assumption $d(\tilde{A},\tilde{B})<3$. Therefore, the vectors $N^*N\xi$, ξ and $N\xi$ are linearly dependent. That is, there is a nontrivial relation $\lambda N^*N\xi+\mu N\xi+\varrho\xi=0$. If $\lambda=0$, then ξ is an eigenvector for N. Because $d(\tilde{A},\tilde{B})\neq 0$, this is not possible. Thus we can assume without loss of generality that $\lambda=1$. From the spectral theorem (recall that N is normal) we conclude that $\xi\in G(\mathscr{E})$, where $\mathscr{E}:=\{z\in\mathbf{C}_1\colon |z|^2+\mu z+\varrho=0\}$.

The next step is to show that $\mu=0$. Assume the contrary, that is, $\mu\neq 0$. Then $\mathscr{E}$ is the intersection of a circle and a straight line. Hence $\mathscr{E}$ consists of at most two points. Therefore, $\xi=\xi_1+\xi_2$, where $N\xi_1=z_1\xi_1$ and $N\xi_2=z_2\xi_2$ with $\xi_1,\xi_2\in\mathscr{H}$ and $z_1,z_2\in\mathbf{C}_1$. Since ξ is not an eigenvector as noted above, it follows that $\xi_1\neq 0$, $\xi_2\neq 0$ and $z_1\neq z_2$. From this we conclude that $\mathscr{K}=\mathrm{Lin}\,\{\xi,N\xi\}=\mathrm{Lin}\,\{\xi_1,\xi_2\}$ and $\dim\mathscr{K}=2$. Since ξ_1 and ξ_2 are eigenvectors of $X=\mathrm{Re}\,N$ as well, $\tilde{X}=$ $=X-(1/2)N\xi\otimes\xi-(1/2)\xi\otimes N\xi$ leaves $\mathscr{K}$ invariant. Since $\mathscr{N}(X)=\{0\}$ by assumption, $\tilde{X}$ maps $\mathscr{K}$ onto $\mathscr{K}$. Therefore, $\mathscr{R}(\tilde{X})\supseteqq\mathscr{K}\supseteqq\mathscr{R}([\tilde{X},\tilde{Y}])$ and $\mathscr{R}(\tilde{X})\cap$ $\cap\mathscr{R}([\tilde{X},\tilde{Y}])\neq\{0\}$. Because of (4), this is the desired contradiction. This proves that $\mu=0$.

By the preceding, $\mathscr{E}=\{z\in\mathbf{C}_1\colon |z|^2=-\varrho\}$. Since $\mathscr{E}$ contains more than one point (otherwise $\xi=0$ or ξ is an eigenvector for N), it follows that $-\varrho$ is positive. Let $r:=\sqrt{-\varrho}$. Let $\mathscr{H}_r=G(\mathscr{E})\mathscr{H}\equiv G(K_r)\mathscr{H}$. Since $\xi\in\mathscr{H}_r$, $2i[\tilde{X},\tilde{Y}]=-\varrho\xi\otimes\xi+$ $+N\xi\otimes N\xi$. From this we easily see that the case $d(\tilde{A},\tilde{B})=1$ is not possible. Indeed, $d(\tilde{A},\tilde{B})=\dim\mathscr{R}([\tilde{X},\tilde{Y}])=1$ implies that ξ is an eigenvector for N. But this leads to $d(\tilde{A},\tilde{B})=0$.

We show that $(Y-a)\xi\notin\mathscr{R}(X)$ for all $a\in\mathbf{R}_1$. Otherwise, $(Y-a)\xi=X\eta$ for some $\eta\in\mathscr{H}$. Then $N\xi-ai\xi=i(Y-a)\xi+X\xi=X\psi$ with $\psi:=i\eta+\xi$ and $\tilde{X}\psi=$ $=X\psi-(1/2)\langle\psi,N\xi\rangle\xi-(1/2)\langle\psi,\xi\rangle N\xi=(1-(1/2)\langle\psi,\xi\rangle)N\xi-(ai+(1/2)\langle\psi,N\xi\rangle)\xi\in\mathscr{K}$. Because $\xi\in\mathscr{H}_r$, $\dim\mathscr{K}\leqq 2$. Since $\mathscr{R}([\tilde{X},\tilde{Y}])\subseteq\mathscr{K}$ and we are in the case $d(\tilde{A},\tilde{B})=2$, we obtain $\mathscr{K}=\mathscr{R}([\tilde{X},\tilde{Y}])$. Since $\mathscr{R}(\tilde{X})\cap\mathscr{R}([\tilde{X},\tilde{Y}])=\{0\}$ and $\mathscr{N}(\tilde{X})=\{0\}$ by (4), it follows that $\tilde{X}\psi=0$ and hence $\psi=0$. Putting this in the above formula for $\tilde{X}\psi$, we get $\tilde{X}\psi=0=N\xi-ai\xi=0$ which contradicts $d(\tilde{A},\tilde{B})\neq 0$. This proves that $(Y-a)\xi\notin\mathscr{R}(X)$. Obviously, $\mathscr{H}_r$ reduces both X and Y. Because $\xi\in\mathscr{H}_r$, $(Y-a)X^{-1}\restriction\mathscr{H}_r$ must be unbounded for all $a\in\mathbf{R}_1$. The proof for $(X-a)Y^{-1}\restriction\mathscr{H}_r$ is almost the same.

We now turn to the proof of the opposite directions. We begin with (i).

Assume that the operators $(X-a)(X-b)Y^{-1}$ and $(Y-a)(Y-b)X^{-1}$ are unbounded for all $a,b\in\mathbf{R}_1$. Suppose for a moment we have proved the existence of a vector $\xi\in\mathscr{H}$ of norm one such that

(11) $\quad (X-z)(X-w)\xi\notin\mathscr{R}(Y)$ and $(Y-z)(Y-w)\xi\notin\mathscr{R}(X)$ for all $z,w\in\mathbf{C}_1$.

Since X and Y are bounded and $XY=YX$, this implies that

(12) $$(z_1X^2+z_2X+z_3)\xi\notin\mathscr{R}(Y) \quad\text{and}\quad (z_1Y^2+z_2Y+z_3)\xi\notin\mathscr{R}(X)$$
$$\text{for all } z_1,z_2,z_3\in\mathbf{C}_1,\quad (z_1,z_2,z_3)\neq 0.$$

We check that $\tilde{X}=\operatorname{Re}N(I-E_\xi)$ and $\tilde{Y}=\operatorname{Im}N(I-E_\xi)$ satisfy condition (4). Recall that $\mathscr{R}([\tilde{X},\tilde{Y}])\subseteq\mathscr{K}$. Suppose that $\tilde{X}\varphi\in\mathscr{K}$, that is,

$$\tilde{X}\varphi=X\varphi-(1/2)\langle\varphi,N\xi\rangle\xi-(1/2)\langle\varphi,\xi\rangle N\xi=(\lambda+\mu N+\varrho N^*N)\xi$$
$$\text{for some }\lambda,\mu,\varrho\in\mathbf{C}_1.$$

Then $\quad X(\varphi-((1/2)\langle\varphi,\xi\rangle+\mu)\xi-\varrho X\xi)=(\lambda+(1/2)\langle\varphi,N\xi\rangle)\xi+(\mu+(1/2)\langle\varphi,\xi\rangle)iY\xi+$ $+\varrho Y^2\xi=:\psi$. By (12), $\psi=0$. Again by (12), the vectors ξ, $Y\xi$ and $Y^2\xi$ are linearly independent. Hence $(1/2)\langle\varphi,\xi\rangle+\mu=\varrho=0$ which gives $X\varphi=0$. Since $\mathscr{N}(X)=\{0\}$ by assumption, $\varphi=0$. This proves $\mathscr{R}(\tilde{X})\cap\overline{\mathscr{R}([\tilde{X},\tilde{Y}])}=\{0\}$. The same argument shows that $\mathscr{R}(\tilde{Y})\cap\overline{\mathscr{R}([\tilde{X},\tilde{Y}])}=\{0\}$ and $\mathscr{N}(\tilde{X})=\mathscr{N}(\tilde{Y})=\{0\}$. By Theorem 1.7, $\{\tilde{A}:=\tilde{X}^{-1},\tilde{B}:=\tilde{Y}^{-1}\}\in\mathfrak{N}_1$. It remains to prove that $d(\tilde{A},\tilde{B})=3$. As noted in the proof of the necessity part, it suffices to show that ξ, $N\xi$ and $N^*N\xi$ are linearly independent. If $\lambda\xi+\mu N\xi+\varrho N^*N\xi=0$ for $\lambda,\mu,\varrho\in\mathbf{C}_1$, then $X(\mu\xi+\varrho X\xi)=$ $=-\lambda\xi-\mu iY\xi-\varrho Y^2\xi\in\mathscr{R}(X)$. By (12), this leads to $\lambda=\mu=\varrho=0$.

To complete the proof of (i), we have to prove the existence of a unit vector $\xi\in\mathscr{H}$ satisfying (11). We let $\mathscr{X}$ and $\mathscr{Y}$ denote the x-axis, resp., y-axis. The following preparatory construction will be needed below. Let s be one of the numbers $0,1,2$. Let $\gamma\in\mathbf{R}_1$, and let $\varepsilon>0$. Suppose that $(0,\gamma)\in\sigma(N)$. Since $\mathscr{N}(X)=$ $=\mathscr{N}(Y)=\{0\}$, there exists a sequence $(x_n,y_n)\in\sigma(N)$, $n\in\mathbf{N}$, such that $\lim\limits_n(x_n,y_n)=$ $=(0,\gamma)$ and $x_n\neq 0$ and $y_n\neq 0$ for all $n\in\mathbf{N}$. Let us assume in addition that

$\lim_n |y_n-\gamma|^s|x_n|^{-1}=\infty$. Let $\mathscr{F}_\gamma := \bigcup_{n\geqq k} U_{\varepsilon_n}(x_n, y_n)$, where $0<2\varepsilon_n<\min\{|x_n|, |y_n|\}$ and k is chosen so large that $|y_n-\gamma|^s>2|x_n|$ for $n\geqq k$ and $\mathscr{F}_\gamma\subseteq U_\varepsilon(0,\gamma)$. Then $\overline{\mathscr{F}}_\gamma\cap\mathscr{Y}=\{(0,\gamma)\}$. First we consider the case where $\gamma\neq 0$ or $s\neq 0$. Since $(Y-\gamma)^sX^{-1}$ is obviously unbounded on $G(\mathscr{F}_\gamma)$, there is a vector $\varphi_\gamma\in G(\mathscr{F}_\gamma)$ such that $\varphi_\gamma\notin\mathscr{D}((Y-\gamma)^sX^{-1})$, that is, $(Y-\gamma)^s\varphi_\gamma\notin\mathscr{R}(X)$. Now let $\gamma=0$ and $s=0$. Since $\overline{\mathscr{F}}_0\cap\mathscr{X}=\overline{\mathscr{F}}_0\cap\mathscr{Y}=\{(0,0)\}$ by construction and the function $f(x,y)=\min\{1/|x|, 1/|y|\}$ is not essentially bounded on $\mathscr{F}_0$ w.r.t. the spectral measure $G(\cdot)$, we can find a vectoı $\varphi_0\in G(\mathscr{F}_0)$ such that $\varphi_0\notin\mathscr{D}(f(X,Y))$. By the spectral theorem this implies $Y\varphi_0\notin\mathscr{R}(X)$ and $X\varphi_0\notin\mathscr{R}(Y)$.

In order to construct ξ we consider three cases.

Case 1: $\mathscr{Y}\cap\sigma(N)$ contains at least three different points $(0,\gamma_1), (0,\gamma_2), (0,\gamma_3)$. We apply the preceding construction to each γ_j in case $s=0, 0<3\varepsilon<\min\{|\gamma_j-\gamma_1|; j\neq 1\}$ and we obtain vectors $\varphi_{\gamma_1}, \varphi_{\gamma_2}, \varphi_{\gamma_3}$. Set $\xi_1=\varphi_{\gamma_1}+\varphi_{\gamma_2}+\varphi_{\gamma_3}$.

Case 2: $\mathscr{Y}\cap\sigma(N)$ consists of two different points $(0,\gamma_1), (0,\gamma_2)$.

Since $(Y-\gamma_1)(Y-\gamma_2)X^{-1}$ is unbounded and $\mathscr{N}(X)=\mathscr{N}(Y)=\{0\}$, there is a sequence $(x_n, y_n)\in\sigma(N)$ such that $\lim_n|(y_n-\gamma_1)(y_n-\gamma_2)x_n^{-1}|=\infty$ and $x_n\neq 0, y_n\neq 0$ for all $n\in\mathbf{N}$. By passing to a subsequence if necessary (recall that (y_n) is bounded, since Y is bounded) we can assume that $\lim_n y_n=:\gamma$ exists. Since $(0,\gamma)\in\mathscr{Y}\cap\sigma(N)$, we must have $\gamma=\gamma_1$ or $\gamma=\gamma_2$. Say $\gamma=\gamma_1$. Since $\gamma_1\neq\gamma_2$ and Y is bounded, we have $\lim_n x_n=0$. Let $0<3\varepsilon<|\gamma_1-\gamma_2|$. Setting $s=1$ in case γ_1 and $s=0$ in case γ_2, the above construction yields vectors φ_{γ_1} and φ_{γ_2}. Put $\xi_1=\varphi_{\gamma_1}+\varphi_{\gamma_2}$.

Case 3: $\mathscr{Y}\cap\sigma(N)$ contains only one point $(0,\gamma)$.

Because $(Y-\gamma)^2X^{-1}$ is unbounded, we can find a sequence $(x_n, y_n)\in\sigma(N)$ such that $\lim_n|(y_n-\gamma)^2x_n^{-1}|=\infty$ and $x_n\neq 0, y_n\neq 0$ for $n\in\mathbf{N}$. As in Case 2 we can assume without loss of generality that $\lim_n y_n=\gamma$ and $\lim_n x_n=0$. We apply the above construction in case $s=2, \varepsilon=1$ and we set $\xi_1=\varphi_\gamma$.

Since X^{-1} must be unbounded, $\mathscr{Y}\cap\sigma(N)\neq\emptyset$. That is, we have discussed all possible cases.

It is not difficult to see that in each case $(Y-z)(Y-w)\xi_1\notin\mathscr{R}(X)$ for all $z, w\in\mathbf{C}_1$. We check this in Case 2. Recall that, by the spectral theorem, $(Y-z)(Y-w)\xi_1\notin\mathscr{R}(X)$ is equivalent to $\iint|(y-z)(y-w)x^{-1}|^2\, d\|G(x,y)\xi_1\|^2=\infty$. First let $z\neq\gamma_2$ and $w\neq\gamma_2$. Since $\overline{\mathscr{F}}_{\gamma_2}\cap\mathscr{Y}=\{(0,\gamma_2)\}$, we then have $(Y-z)(Y-w)\varphi_{\gamma_2}\notin\mathscr{R}(X)$. Because $U_\varepsilon(0,\gamma_1)\cap U_\varepsilon(0,\gamma_2)=\emptyset$, $(Y-z)(Y-w)\xi_1\notin\mathscr{R}(X)$. Now let $z=\gamma_2$. It is plane from the construction of φ_{γ_1} that $(Y-w)\varphi_{\gamma_1}\notin\mathscr{R}(X)$ for all $w\in\mathbf{C}_1$. Again by $U_\varepsilon(0,\gamma_1)\cap\cap U_\varepsilon(0,\gamma_2)=\emptyset$, this gives $(Y-z)(Y-w)\varphi_{\gamma_1}\notin\mathscr{R}(X)$ and $(Y-z)(Y-w)\xi_1\notin\mathscr{R}(X)$.

We now change the role of X and Y and we repeat the same procedure. The corresponding vectors will be denoted by ψ_{δ_j} and ξ_2. If $(0,0)\in\sigma(N)$ and if the case $s=0, \gamma=\delta=0$ occurs in the first *and* in the second procedure, then we set $\psi_0=\varphi_0$. As in the first part, we have $(X-z)(X-w)\xi_2 \notin \mathscr{R}(Y)$ for all $z, w \in \mathbf{C}_1$. If we take the radii of the circles around $(0, \gamma_j)$, resp., $(\delta_1, 0)$ small enough, then except from the possible case $\psi_0=\varphi_0$ we have just mentioned the vectors φ_{γ_j} and ψ_{δ_i} have disjoint support w.r.t. the spectral measure $G(\cdot)$. Therefore, $\xi:= :=(\xi_1+\xi_2)/\|\xi_1+\xi_2\|$ has the desired properties. Now the proof of (i) is complete.

We only sketch the proof of the sufficiency part of (ii). Assume that the operators $(X-a)Y^{-1} \restriction \mathscr{H}_r$ and $(Y-a)X^{-1} \restriction \mathscr{H}_r$ are unbounded for all $a \in \mathbf{R}_1$. Since $\mathscr{H}_r=G(K_r)\mathscr{H}$ reduces X and Y, we can assume for simplicity in notation that $\mathscr{H}=\mathscr{H}_r$. Then $\sigma(N) \subseteqq K_r$. Since $(X-r)Y^{-1}$ is unbounded, there are points $(x_n, y_n) \in \sigma(N)$, $n \in \mathbf{N}$, so that $\lim_n |(x_n-r)y_n^{-1}|=\infty$. By taking a subsequence if necessary, we can assume that $\lim_n x_n=:\gamma$ exists. Since $x_n^2+y_n^2=r$ for $n \in \mathbf{N}$, it follows that $\gamma=-r, \lim_n y_n=0$ and $(-r, 0) \in \sigma(N)$. Using that $(X+r)Y^{-1}$, $(Y-r)X^{-1}$ and $(Y+r)X^{-1}$ are unbounded, the same argument shows that $(r, 0), (0, -r), (0, r) \in \sigma(N)$. Hence we can take vectors ξ_1, ξ_2, ξ_3 and ξ_4 in $\mathscr{H}(=\mathscr{H}_r)$ supported in the neighbourhood of $(-r, 0), (r, 0), (0, -r)$, resp., $(0, r)$ w.r.t. $G(\cdot)$ such that $\xi_1 \notin \mathscr{R}(Y), \xi_2 \notin \mathscr{R}(Y)$ and $\xi_3 \notin \mathscr{R}(X), \xi_4 \notin \mathscr{R}(X)$. Setting $\xi= =\xi_1+\xi_2+\xi_3+\xi_4$, we then have that $(\lambda Y+\mu)\xi \notin \mathscr{R}(X)$ and $(\lambda X+\mu)\xi \notin \mathscr{R}(Y)$ for all $\lambda, \mu \in \mathbf{C}_1$, $(\lambda, \mu) \neq 0$. As in part (i) we can show that X and Y fulfil condition (4) in Theorem 1.7 and $d(\tilde{A}, \tilde{B})=2$.

Now the proof of Theorem 1 is complete.

4.2. Remarks. 1. There are many examples of operators N satisfying the assumptions of Theorem 1. We mention only two of them.

Example 1. Let N be a normal operator such that $\mathscr{N}(\operatorname{Re} N)=\mathscr{N}(\operatorname{Im} N)= =\{0\}$. If $\sigma(N)$ intersects both the x-axis and the y-axis in at least three points, then (as we have seen in the preceding proof) the assumptions of part (i) are fulfilled. Hence, by Theorem 1, ξ can be chosen such that the corresponding pair $\{\tilde{A}, \tilde{B}\}$ is in $\mathfrak{N}_1$ and has defect number three.

Example 2. Let R be an unbounded self-adjoint operator, and let $N= =(R-i)(R+i)^{-1}$ be its Cayley transfoım. N is unitary. Suppose that $\mathscr{N}(R)= =\mathscr{N}(R+I)=\mathscr{N}(R-I)=\{0\}$. Obviously, this is equivalent to $\mathscr{N}(\operatorname{Re} N)= =\mathscr{N}(\operatorname{Im} N)=\{0\}$. If the points $0, -1$ and 1 are in $\sigma(R)$, then the assumptions of part (ii) are satisfied (in case $r=1$) and Theorem 1 (ii), yields a pair $\{\tilde{A}, \tilde{B}\} \in \mathfrak{N}_1$ with $d(\tilde{A}, \tilde{B})=2$. In this case $N(I-E_\xi)$ is a partial isometry with corank one and defect one.

2. We want to interpret the method used in this section from another point of view. Again let $N=X+iY$ be normal and assume that $\mathscr{N}(X)=\mathscr{N}(Y)=\{0\}$. Of course, $\{A:=X^{-1}, B:=Y^{-1}\} \in \mathfrak{N}_1$ and $d(A, B)=0$. Suppose that $\{\tilde{A}=\tilde{X}^{-1}$,

$\tilde{B}=\tilde{Y}^{-1}\}$ is a pair of the class $\mathfrak{N}_1$ constructed as in Theorem 1 (i) or (ii). Then $\tilde{X}=X-(1/2)N\xi\otimes\xi-(1/2)\xi\otimes N\xi$ and $\tilde{Y}=Y-(1/2)iN\xi\otimes\xi+(1/2)i\xi\otimes N\xi$. We denote by F the orthogonal projection on $\mathscr{H}_\xi:=\mathrm{Lin}\,\{\xi, X\xi, Y\xi, NX\xi, NY\xi\}$. Modifying some arguments of the proof of Theorem 1, it can be shown that $\mathscr{R}(XY)\cap\mathscr{H}_\xi=\{0\}$. This implies that $\mathscr{D}_0:=XY(I-F)\mathscr{H}$ is dense in $\mathscr{H}$. Since the vectors ξ and $N\xi$ are orthogonal on $(I-F)\mathscr{H}, X(I-F)\mathscr{H}$ and $Y(I-F)\mathscr{H}$ by construction, it is easily seen that $A\restriction\mathscr{D}_0=\tilde{A}\restriction\mathscr{D}_0, B\restriction\mathscr{D}_0=\tilde{B}\restriction\mathscr{D}_0$ and $A\restriction B\mathscr{D}_0=\tilde{A}\restriction B\mathscr{D}_0$, $B\restriction A\mathscr{D}_0=\tilde{B}\restriction A\mathscr{D}_0$. In other words, the pair $\{\tilde{A},\tilde{B}\}\in\mathfrak{N}_1$ can be considered as an extension of the restriction to the dense domain $\mathscr{D}_0\subseteq\mathscr{D}(AB)\cap\mathscr{D}(BA)\cap\mathscr{D}(\tilde{A}\tilde{B})\cap\cap\mathscr{D}(\tilde{B}\tilde{A})$ of the strongly commuting pair $\{A,B\}\in\mathfrak{N}_1$.

The author would like to thank Kevin Clancey for valuable discussions during the eighth conference on operator theory in Romania, 1982.

References

[1] N. Bourbaki, *Intégration*, Hermann (Paris, 1967).

[2] R. Carey and J. D. Pincus, Construction of seminormal operators with prescribed mosaic, *Indiana Univ. Math. J.*, **23** (1974), 1155—1165.

[3] K. Clancey, *Seminormal operators*, Lecture Notes in Mathematics 742, Springer-Verlag (Berlin, 1979).

[4] K. Clancey, Toeplitz models for operators with one dimensional self-commutators, in: *Proceedings of the 8th conference on operator theory in Timisoara and Herculane, 1982*, OT 11, Birkhäuser (Basel, 1983).

[5] R. G. Douglas, H. S. Shapiro and A. L. Shields, Cyclic vectors and invariant subspaces of the backward shift operator, *Ann. Inst. Fourier*, **20** (1970), 37—76.

[6] B. Fuglede, On the relation $PQ-QP=-iI$, *Math. Scand.*, **20** (1967), 79—88.

[7] P. R. Halmos, *A Hilbert space problem book*, D. Van Nostrand Co. (Princeton, New Jersey, 1967).

[8] J. W. Helton and R. Howe, *Integral operators, commutator traces, index and homology*, Lecture Notes in Mathematics 345, Springer-Verlag (Berlin, 1973).

[9] P. E. T. Jørgensen and R. T. Moore, Derivation on modules of unbounded operators with applications to partial differential operators on Riemann surfaces, Preprint No. 27, Aarhus University, 1980/81.

[10] E. Nelson, Analytic vectors, *Ann. Math.*, **70** (1959), 572—615.

[11] J. D. Pincus, Commutators and systems of integral equations. I, *Acta Math.*, **121** (1968), 219—249.

[12] C. R. Putnam, *Commutation properties of Hilbert space operators and related topics*, Springer-Verlag (Berlin, 1967).

[13] R. T. Powers, Self-adjoint algebras of unbounded operators, *Comm. Math. Phys.*, **21** (1971), 85—124.

[14] F. Riesz und B. Sz.-Nagy, *Vorlesungen über Funktionalanalysis*, DVW (Berlin, 1956).

[15] K. Schmüdgen, Unpublished manuscript, Leipzig, 1979.

[16] Xa Dao-xeng, On non-normal operators, *Chinese Math.*, **3** (1963), 232—246.

[17] J. von Neumann, Zur Theorie der unbeschränkten Matrizen, *J. Reine Angew. Math.*, **161** (1929), 208—236.

SEKTION MATHEMATIK
KARL-MARX-UNIVERSITÄT LEIPZIG
KARL-MARX PLATZ
7010 LEIPZIG, GERMAN DEMOCRATIC REPUBLIC

Acta Sci. Math., 47 (1984), 147—167

A unified approach to fundamental theorems of approximation by sequences of linear operators and their dual versions

S. RIES*) and R. L. STENS

Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday on 29. July 1983, in high esteem

1. Introduction

The theorems of Jackson and Bernstein as well as those of Stečkin and Zamansky and their converses play a fundamental role in the theory of best approximation for periodic functions by trigonometric polynomials. These results have been generalized to the setting of abstract Banach spaces by Butzer and Scherer [7], [8], [9], who also proved corresponding approximation theorems for sequences of commutative bounded linear operators $\{T_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} T_n f = f$ in the norm topology as well as so called Jackson and Bernstein-type inequalities (cf. [10]).

This paper is concerned with the latter aspect. The aim is to weaken the assumption upon the sequence $\{T_n\}$ in the sense that $\lim_{n\to\infty} T_n f = f$ needs to hold only in a certain weak topology. This enables one to handle sequences of operators converging in the usual weak or weak* topology towards the identity operator.

Our main theorem (Theorem 1) subsumes not only the Butzer—Scherer theorem (Theorem 2) mentioned above, but also the corresponding results for sequences of dual operators (Theorem 3), contained in [17], [18]. This theory can be applied to classical linear approximation processes such as summation methods for Fourier series or semigroups of bounded linear operators. The further advantage is that it enables one to investigate processes defined by sequences of the dual operators.

Received November 2, 1982.

*) This author was supported by grant no. Ne 171/5—1 of the Deutsche Forschungsgemeinschaft.

The paper is divided as follows. After a preliminary section containing definitions and elementary lemmas the general approximation theorem is established in Section 3. From this result the Butzer—Scherer theorem is deduced in Section 4. Its dual version together with applications to convolution integrals of periodic functions are dealt with in Section 5.

2. Preliminaries

Let us begin with some basic definitions and results concerning norm-determining sets and linear approximation processes.

Definition 1. Let X be a normed linear space with norm $\|\cdot\|_X$, and let X' denote its dual endowed with the usual norm $\|\cdot\|_{X'}$. For a linear subspace $\mathcal{M}$ in X' the characteristic $\nu(\mathcal{M})$ is defined as

$$\nu(\mathcal{M}) := \inf\{p_{\mathcal{M}}(f);\ f\in X,\ \|f\|_X = 1\} \tag{2.1}$$

where $p_{\mathcal{M}}(f)$ is given by

$$p_{\mathcal{M}}(f) := \sup\{|f'(f)|;\ f'\in\mathcal{M},\ \|f'\|_{X'} = 1\} \quad (f\in X). \tag{2.2}$$

If $\nu(\mathcal{M})>0$, then $\mathcal{M}$ is said to be norm-determining (for X).

It follows from the inequalities

$$\nu(\mathcal{M})\|f\|_X \leqq p_{\mathcal{M}}(f) \leqq \|f\|_X \quad (f\in X) \tag{2.3}$$

that $\|\cdot\|_X$ and $p_{\mathcal{M}}(\cdot)$ are equivalent norms for X, provided $\mathcal{M}$ is norm-determining. If, in addition, $\mathcal{M}$ is closed in X', then one has

Lemma 1 (cf. [19, p. 203]). *Let F be a subset of a normed linear space X, and let $\mathcal{M}$ be a closed norm-determining subspace of X'. If* $\sup\{|f'(f)|; f\in F\}<\infty$ *for each $f'\in\mathcal{M}$; then* $\sup\{\|f\|_X; f\in F\}<\infty$.

Next we introduce the concept of $\mathcal{M}$-weak convergence in X.

Definition 2. Let $\mathcal{M}$ be a norm-determining set for the linear space X. A sequence $\{f_n\}_{n=1}^{\infty}$ in X is said to be $\mathcal{M}$-weakly convergent to $f\in X$ ($\mathcal{M}$-$\lim_{n\to\infty} f_n=f$), if

$$\lim_{n\to\infty}\langle f', f_n-f\rangle = 0 \quad (f'\in\mathcal{M}).$$

Note that the $\mathcal{M}$-limit is uniquely determined, since $f'(f)=0$ for all $f'\in\mathcal{M}$ implies $f=0$ by (2.3). Moreover, if $\mathcal{M}$ is closed in X', then Lemma 1 yields that every $\mathcal{M}$-weak convergent sequence is bounded.

Choosing $\mathcal{M}=X'$ gives $\nu(X')=1$, and it follows that weak convergence is a particular case of $\mathcal{M}$-weak convergence. One may also take $\mathcal{M}=J(X)$, where

J is the canonical mapping from X into X''. Again one has $\nu(J(X))=1$, and $\mathcal{M}$-weak convergence in X' turns out to be w^*-convergence (cf. [19, pp. 208, 209]).

Let us finally mention that the only closed norm-determining subspace for a reflexive Banach space X is X' itself. This follows from the facts that a closed subspace of X' is also w^*-closed (cf. [13, p. 422]) and, on the other hand, a norm-determining subspace must be w^*-dense in X'. For further properties of norm-determining sets and $\mathcal{M}$-weak convergence see [1], [12], [19, Sec. 4.4].

Now we consider sequences of bounded linear operators from X into itself converging $\mathcal{M}$-weakly towards the identity.

Definition 3. Let X be a normed linear space, $\mathcal{M}$ a closed norm-determining subspace of X', and $\{T_n\}_{n=1}^{\infty}$ a sequence of bounded linear operators mapping X into itself with the properties

$$T_n T_m f = T_m T_n f \quad (f \in X;\ m, n \in \mathrm{N} := \{1, 2, \ldots\}), \tag{2.4}$$

$$\mathcal{M}\text{-}\lim_{n\to\infty} T_n f = f \quad (f \in X). \tag{2.5}$$

Then $\{T_n\}$ is called a commutative, $\mathcal{M}$-weak linear approximation process on X ($\mathcal{M}$-LAP).

Note that $\mathcal{M}$ will be assumed to be a closed norm-determining subspace of X' when speaking of an $\mathcal{M}$-LAP on X.

For an $\mathcal{M}$-LAP on a Banach space X the following inequalities hold; the first is a generalization of the well known uniform boundedness principle for sequences of strongly convergent operators.

Lemma 2. *Let $\{T_n\}_{n=1}^{\infty}$ be an $\mathcal{M}$-LAP on a Banach space X. Then*

$$\|T_n f\|_X \leqq M \|f\|_X{}^{1)} \quad (f \in X;\ n \in \mathrm{N}), \tag{2.6}$$

$$\|T_n f - f\|_X \leqq \frac{1}{\nu(\mathcal{M})} \sum_{k=0}^{\infty} \|T_{n2^k} f - T_{n2^{k+1}} f\|_X \quad (f \in X;\ n \in \mathrm{N}). \tag{2.7}$$

Proof. For fixed $f \in X$ the sequence $\{T_n f\}_{n=1}^{\infty}$ is an $\mathcal{M}$-weakly convergent sequence in X, and it follows by Lemma 1 that $\|T_n f\|_X \leqq M_f$, where M_f depends on f but not on n. Since X is a Banach space one can apply the classical uniform boundedness principle to deduce (2.6). Concerning (2.7), one has for $f' \in X', f \in X$ that

$$\langle f', T_n f - f\rangle = \sum_{k=0}^{\infty} \langle f', T_{n2^k} f - T_{n2^{k+1}} f\rangle.$$

[1]) Throughout M denotes a positive constant, the value of which may be different at each occurrence, even in a given line. M is always independent of the quantities on the right margin.

Then (2.3) and (2.2) yield the assertion since

$$\nu(\mathscr{M})\|T_nf-f\|_X \leqq p_{\mathscr{M}}(T_nf-f) =$$

$$= \sup\left\{\left|\sum_{k=0}^{\infty}\langle f', T_{n2^k}f-T_{n2^{k+1}}f\rangle\right|;\ f'\in\mathscr{M},\ \|f'\|_{X'}=1\right\} \leqq$$

$$\leqq \sum_{k=0}^{\infty}\|T_{n2^k}f-T_{n2^{k+1}}f\|_X.$$

3. Order of approximation for $\mathscr{M}$-LAP's

In this section we consider $\mathscr{M}$-LAP's converging strongly towards the identity on certain subsets of X, and investigate the rate of convergence.

3.1. K-functional; Jackson and Bernstein-type inequalities

Definition 4. Let X be a linear space and Y a linear subspace of X with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The K-functional between X and Y is defined as

$$K(t,f;\ X,Y) := \inf\{\|f_1\|_X+t\|f_2\|_Y;\ f=f_1+f_2,\ f_1\in X,\ f_2\in Y\}\quad (f\in X;\ t>0).$$

For fixed $f\in X$ the K-functional is nondecreasing on $(0,\infty)$, and satisfies

$$K(\lambda t,f;\ X,Y) \leqq \max\{1,\lambda\}K(t,f;\ X,Y)\quad (\lambda, t>0). \tag{3.1}$$

Furthermore, $\lim_{t\to 0+} K(t,f;\ X,Y)=0$ for each $f\in X$ if and only if Y is dense in X.

Definition 5. Let X, Y be given as in Definition 4, $\mathscr{M}$ be a closed, norm-determining subspace of X', and $\{T_n\}_{n=1}^{\infty}$ an $\mathscr{M}$-LAP on X. If for some $\alpha\geqq 0$ there holds

$$\|T_ng-g\|_X \leqq Mn^{-\alpha}\|g\|_Y\quad (g\in Y;\ n\in\mathbf{N}), \tag{3.2}$$

then $\{T_n\}$ is said to satisfy a Jackson-type inequality of order α on X with respect to Y.

If $T_nf\in Y$ for all $f\in X$, $n\in\mathbf{N}$ and

$$\|T_nf\|_Y \leqq Mn^{\alpha}\|f\|_X\quad (f\in X;\ n\in\mathbf{N}), \tag{3.3}$$

then $\{T_n\}$ is said to satisfy a (weak) Bernstein-type inequality of order α on X with respect to Y.

In contrast to (3.3) one would speak of a *strong* Bernstein-type inequality, if $T_n f \in Y$ for all $f \in X$ and

$$\|g\|_Y \leqq M n^\alpha \|g\|_X \quad (g \in T_n(X);\ n \in \mathbf{N}). \tag{3.4}$$

In this respect see also the remarks following Theorem 1. Finally we need

Definition 6. The class of positive, nondecreasing functions φ defined on $(0, 1]$ with $\lim_{t\to 0+} \varphi(t)=0$ and $\varphi(1)<\infty$ is denoted by Φ.

The following condition on $\varphi \in \Phi$, $\alpha, \beta \geqq 0$ will be of interest, namely

$$\sum_{1\leqq 2^j \leqq t^{-1}} 2^{\alpha j} \varphi(2^{-j}) = O(t^{-\alpha}\varphi(t)) \quad (t \to 0+), \tag{3.5}$$

$$\sum_{j=0}^{\infty} (t^{-1}2^j)^\beta \varphi(t2^{-j}) = O(t^{-\beta}\varphi(t)) \quad (t \to 0+). \tag{3.6}$$

For conditions which are equivalent to (3.5) or (3.6) we refer to the appendix.

Lemma 3. *If* $\varphi \in \Phi$, $\alpha, \beta \geqq 0$, *then* (3.5) *implies* $\lim_{t\to 0+} t^{-\alpha}\varphi(t)=\infty$, *and* (3.6) *implies* $\lim_{t\to 0+} t^{-\beta}\varphi(t)=0$. *In particular, if* (3.5) *and* (3.6) *are valid, then* $\alpha>\beta$. *Furthermore,* (3.5) *implies*

$$\sum_{j=0}^{k} 2^{\alpha j} \varphi(2^{-j}) = O(2^{\alpha k}\varphi(2^{-k})) \quad (k \to \infty). \tag{3.7}$$

Proof. One has by (3.5) for $t=2^{-k}$ that $\varphi(1)\leqq M2^{\alpha k}\varphi(2^{-k})$. This yields, again by (3.5),

$$(m+1)\varphi(1) \leqq M \sum_{k=0}^{m} 2^{\alpha k}\varphi(2^{-k}) \leqq M 2^{\alpha m}\varphi(2^{-m}) \quad (m \in \mathbf{N}),$$

giving $\lim_{m\to\infty} 2^{\alpha m}\varphi(2^{-m})=\infty$. The first assertion now follows by choosing $m\in\mathbf{N}$ such that $2^{m-1}\leqq t^{-1}<2^m$ since in this case $t^{-\alpha}\varphi(t)>2^{(m-1)\alpha}\varphi(2^{-m})$. Concerning the second part, one has by the convergence of the series in (3.6) that $\lim_{j\to\infty} (2^j m)^\beta \varphi(2^{-j}m^{-1})=0$, at least for m large enough. If one takes $m\in\mathbf{N}$ such that $2^j m \leqq t^{-1} < 2^{j+1}m$, m fixed, then one can complete the proof as before. Finally, if $k \in \mathbf{P} := \{0, 1, 2 \dots\}$ satisfies $2^k \leqq t^{-1} < 2^{k+1}$, then

$$\sum_{j=0}^{k} 2^{\alpha j}\varphi(2^{-j}) = \sum_{1\leqq 2^j\leqq t^{-1}} 2^{\alpha j}\varphi(2^{-j}) \leqq M t^{-\alpha}\varphi(t) \leqq M 2^{\alpha k}\varphi(2^{-k}) \quad (k \in \mathbf{P}),$$

which is (3.7).

3.2. The fundamental theorem for approximation processes. Our main results now read as follows.

Theorem 1. a) *Let X be a Banach space, Y a linear subspace of X and $\mathcal{M}$ a closed norm-determining subspace of X'. Further, let $\{T_n\}_{n=1}^{\infty}$ be an $\mathcal{M}$-LAP satisfying Jackson and Bernstein-type inequalities of order $\alpha>0$ on X with respect to Y, and let $\varphi\in\Phi$ be such that* (3.5) *holds. Then the following assertions are equivalent for $f\in X$:*

(i) $\|T_nf-f\|_X = O(\varphi(n^{-1})) \quad (n\to\infty)$,

(ii) $K(t^\alpha, f;\ X, Y) = O(\varphi(t)) \quad (t\to 0+)$.

b) *Suppose, in addition, that Z is a Banach space continuously embedded in X*[2]) *such that $\{T_n\}$ satisfies Jackson and Bernstein-type inequalities of order $\beta\geqq 0$ on X with respect to Z and assume that* (3.6) *holds for $\varphi\in\Phi$. Then each of the following assertions is equivalent to those of part* a) *for $f\in X$:*

(iii) $f\in Z$ and $\|T_nf-f\|_Z = O(n^\beta\varphi(n^{-1})) \quad (n\to\infty)$,

(iv) $\|T_nf\|_Y = O(n^\alpha\varphi(n^{-1})) \quad (n\to\infty)$.

The proof of this theorem is based on the following four lemmas.

Lemma 4. *If $X, Y, \mathcal{M}$ and $\{T_n\}_{n=1}^{\infty}$ are given as in Theorem 1. a), then*

$$\|T_nf-f\|_X \leqq MK(n^{-\alpha}, f;\ X, Y) \quad (f\in X;\ n\in\mathbf{N}). \tag{3.8}$$

Proof. For each representation $f=f_1+f_2$ with $f_1\in X, f_2\in Y$ one has in view of (2.6) and the Jackson-type inequality (3.2)

$$\|T_nf-f\|_X \leqq \|T_nf_1-f_1\|_X+\|T_nf_2-f_2\|_X \leqq M\|f_1\|_X+n^{-\alpha}\|f_2\|_Y.$$

Taking the infimum over all such representations yields (3.8).

Lemma 5. *Under the assumptions of Theorem 1. a) assertion* (ii) *implies*

$$\|T_nf\|_Y = O(n^\alpha\varphi(n^{-1})) \quad (f\in X;\ n\to\infty).$$

Proof. By (2.4) one has for each $k\in\mathbf{N}$

$$\begin{aligned} T_nf = {} & T_n(f-T_{2^k}f)-T_{2^k}(f-T_nf)+T_1f+ \\ & +\sum_{j=1}^{k}[T_{2^j}(f-T_{2^{j-1}}f)-T_{2^{j-1}}(f-T_{2^j}f)]. \end{aligned} \tag{3.9}$$

[2]) "Z continuously embedded in X" means that Z is a subspace of X and that the identity map is continuous, i.e., in case of normed linear spaces that $\|f\|_X\leqq M\|f\|_Z$ for all $f\in Z$.

So one deduces by the Bernstein-type inequality on X with respect to Y that

$$\begin{aligned}\|T_n f\|_Y \leq M\{n^\alpha \|f-T_{2^k}f\|_X + 2^{\alpha k}\|f-T_n f\|_X + \|f\|_X + \\ + \sum_{j=1}^{k} [2^{\alpha j}\|f-T_{2^{j-1}}f\|_X + 2^{\alpha(j-1)}\|f-T_{2^j}f\|_X]\}.\end{aligned} \tag{3.10}$$

This yields by (ii) the inequality

$$\|T_n f\|_Y \leq M\{n^\alpha \varphi(2^{-k}) + 2^{\alpha k}\varphi(n^{-1}) + \|f\|_X + \sum_{j=1}^{k} [2^{\alpha j}\varphi(2^{-j+1}) + 2^{\alpha(j-1)}\varphi(2^{-j})]\}.$$

Choosing now $k\in\mathbf{N}$ such that $2^{k-1}\leq n<2^k$, and using (3.7) gives

$$\|T_n f\|_Y \leq M\{n^\alpha \varphi(n^{-1}) + \|f\|_X\} \quad (n\in\mathbf{N})$$

which in turn implies the assertion by Lemma 3.

Lemma 6. *Under the assumptions of Theorem 1. b) assertion* (i) *implies* (iii).

Proof. For $n, N\in\mathbf{N}$ one has by (2.4) for $f\in Z$

$$\sum_{k=0}^{N} \|T_{n2^k}f - T_{n2^{k+1}}f\|_Z \leq \sum_{k=0}^{N} \{\|T_{n2^k}(f-T_{n2^{k+1}}f)\|_Z + \|T_{n2^{k+1}}(f-T_{n2^k}f)\|_Z\}.$$

Estimating the terms on the right hand side by the Bernstein-type inequality on X with respect to Z, (i) yields

$$\begin{aligned}&\sum_{k=0}^{N} \|T_{n2^k}f - T_{n2^{k+1}}f\|_Z \leq \\ &\leq M\left\{\sum_{k=0}^{N} (n2^k)^\beta \varphi(n^{-1}2^{-k-1}) + (n2^{k+1})^\beta \varphi(n^{-1}2^{-k})\right\} \leq Mn^\beta \varphi(n^{-1}) \quad (n, N\in\mathbf{N}),\end{aligned} \tag{3.11}$$

the latter estimate follows from (3.6). Since Z is a Banach space this implies in particular that there exists a $g\in Z$ such that

$$\lim_{N\to\infty} \left\|\sum_{k=0}^{N} (T_{n2^k}f - T_{n2^{k+1}}f) - g\right\|_Z = 0 \quad (n\in\mathbf{N}).$$

On the other hand it follows from (i) that

$$\lim_{N\to\infty} \left\|\sum_{k=0}^{N} (T_{n2^k}f - T_{n2^{k+1}}f) - (T_n f - f)\right\|_X = 0 \quad (n\in\mathbf{N}).$$

So one obtains by the continuous embedding of Z in X that $T_n f - f = g\in Z$, yielding $f\in Z$, since $T_n f\in Z$ by the Bernstein-type inequality with respect to Z.

Furthermore, one has by (3.11)

$$\|T_n f-f\|_Z = \left\|\sum_{k=0}^{\infty}(T_{n2^k}f-T_{n2^{k+1}}f)\right\|_Z = O\big(n^{\beta}\varphi(n^{-1})\big) \quad (n\to\infty),$$

proving assertion (iii).

Lemma 7. *Under the assumptions of Theorem 1. b) there holds*

$$\|T_n f\|_Y \leqq Mn^{\alpha-\beta}\|f\|_Z \quad (f\in Z;\ n\in\mathbf{N}). \tag{3.12}$$

Proof. Since Z is a subset of X, one can deduce (3.10) for $f\in Z$ as in the proof of Lemma 5. Applying the Jackson-type inequality on X with respect to Z to the right side of that estimate, yields for each $k\in\mathbf{N}$

$$\|T_n f\|_Y \leqq M\{[n^{\alpha}2^{-\beta k}+2^{\alpha k}n^{-\beta}+\sum_{j=1}^{k}(2^{\alpha j}2^{-\beta(j-1)}+2^{\alpha(j-1)}2^{-\beta j}]\|f\|_Z+\|f\|_X\}.$$

The desired result now follows by choosing $k\in\mathbf{N}$ such that $2^{k-1}\leqq n<2^k$, and the fact that Z is continuously embedded in X.

The inequality (3.12) could be regarded as a Bernstein-type inequality of order $\alpha-\beta$ on Z with respect to Y, if one disregards the general assumptions made in Definition 5 (e.g. one has not necessarily that Y is a subspace of Z or that $\{T_n\}$ is an $\mathcal{M}$-LAP on Z).

Proof of Theorem 1. The implication (ii)$\Rightarrow$(i) follows from Lemma 4. Conversely, let (i) be satisfied. Since $T_n f\in Y$ for all $f\in X$, one has by (i) and Lemma 5 that

$$K(t^{\alpha},f;\ X,Y) \leqq \|f-T_n f\|_X+t^{\alpha}\|T_n f\|_Y \leqq M\{\varphi(n^{-1})+t^{\alpha}n^{\alpha}\varphi(n^{-1})\}$$
$$(t>0;\ n\in\mathbf{N}).$$

This yields (ii) by choosing $n\in\mathbf{N}$ such that $n-1<t^{-1}\leqq n$.

Concerning part b), we proceeded by proving (iv)$\Rightarrow$(i)$\Rightarrow$(iii)$\Rightarrow$(iv). Assume that (iv) holds. Using (2.7) and (2.4) one obtains

$$\|T_n f-f\|_X \leqq \frac{1}{\nu(\mathcal{M})}\sum_{j=0}^{\infty}\|T_{n2^j}f-T_{n2^{j+1}}f\|_X \leqq$$

$$\leqq \frac{1}{\nu(\mathcal{M})}\sum_{j=0}^{\infty}\{\|T_{n2^j}f-T_{n2^{j+1}}(T_{n2^j}f)\|_X+\|T_{n2^{j+1}}f-T_{n2^j}(T_{n2^{j+1}}f)\|_X\}.$$

Estimating the terms in curly brackets by the Jackson-type inequality on X with respect to Y, (iv) gives

$$\|T_n f-f\|_X \leqq M \sum_{j=0}^{\infty} \{(n2^{j+1})^{-\alpha}\|T_{n2^j}f\|_Y+(n2^j)^{-\alpha}\|T_{n2^{j+1}}f\|_Y\} \leqq$$

$$\leqq M \sum_{j=0}^{\infty} \{\varphi(n^{-1}2^{-j})+\varphi(n^{-1}2^{-j-1})\} \leqq Mn^{-\beta} \sum_{j=0}^{\infty} (n2^j)^{\beta}\varphi(n^{-1}2^{-j}).$$

So (i) follows by (3.6). The implication (i)$\Rightarrow$(iii) was established in Lemma 6 and (iii)$\Rightarrow$(iv) can be deduced in the same way as Lemma 5, using (3.12). So the proof is complete.

3.3. Remarks. Note that one can always take $Z=X$ in Theorem 1. b). In this case the Jackson and Bernstein-type inequalities of order $\beta=0$ are obviously valid, and one has the equivalence of (i), (ii) and (iv) under the assumption of part a) provided (3.6) holds for $\beta=0$.

Theorem 1 could also be stated for families of bounded linear operators $\{T_t;\ t\in(0,1]\}$ depending on a continuous parameter t satisfying

$$T_t T_s f = T_s T_t f \quad (f\in X;\ s,t\in(0,1]), \quad \mathcal{M}\text{-}\lim_{t\to 0+} T_t f = f \quad (f\in X)$$

instead of (2.4) and (2.5). In this case one has to replace n everywhere by t^{-1} and $n\to\infty$ by $t\to 0+$. The only slight modification is that one uses condition (3.5) instead of (3.7) in the proof of Lemma 5.

The assumption of the commutativity (2.4) in Theorem 1 can be weakend to

$$T_n T_{2n} f = T_{2n} T_n f \quad (f\in X;\ n\in\mathbf{N}). \tag{3.13}$$

Indeed, only this property was used in the proofs. Moreover, it is possible to establish part a) of Theorem 1 without any commutativity assumption, provided the operators T_n satisfy a strong Bernstein-type inequality (3.4) of order $\alpha>0$ with respect to Y together with $T_n f\in T_m(X)$ for all $n,m\in\mathbf{N}$ with $m\geqq n$. Then one can apply the theory of best approximation to

$$E(f;\ T_n(X)) := \inf\{\|f-g\|_X;\ g\in T_n(X)\} \tag{3.14}$$

to deduce

$$\|T_n f-f\|_X = O(n^{-\sigma}) \Rightarrow E(f;\ T_n(X)) = O(n^{-\sigma}) \Rightarrow K(t^\alpha, f;\ X, Y) = O(t^\sigma).$$

The final step, namely $K(t^\alpha, f; X, Y)=O(t^\sigma)\Rightarrow\|T_n f-f\|_X=O(n^{-\sigma})$, then follows by Lemma 4. The equivalence with assertion (iii) of Theorem 1. a) can also be proved in this frame. A similar approach in case of strong approximation processes (cf. Definition 7) can be found in [7], [8]; for results on best approximation see e.g. [7], [8], [9], [15].

Another method to obtain the equivalence of assertions (i) and (ii) for non-commutative operators is to assume the stability condition

$$\|T_n g\|_Y \leqq M\|g\|_Y \quad (g \in Y;\ n \in \mathbf{N}). \tag{3.15}$$

In this regard see [3], [14].

4. Applications to particular approximation processes

4.1. Strong approximation processes. In this section we apply Theorem 1 to so called strong linear approximation processes as well as to their dual versions.

Definition 7. Let X be a normed linear space, and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of bounded linear operators from X into itself satisfying (2.4) together with

$$\lim_{n\to\infty} \|T_n f - f\|_X = 0 \quad (f \in X). \tag{4.1}$$

Then $\{T_n\}$ is called a commutative strong linear approximation process on X (LAP).

Since (4.1) implies weak convergence of $T_n f$ to f, and weak convergence is a particular case of $\mathcal{M}$-weak convergence with $\mathcal{M} = X'$, one can apply Theorem 1 to deduce

Theorem 2. a) *Let X, Y, φ be given as in Theorem 1. a), and $\{T_n\}_{n=1}^{\infty}$ a* LAP *on X satisfying Jackson and Bernstein-type inequalities of order $\alpha > 0$ on X with respect to Y. Then assertions* (i) *and* (ii) *of Theorem 1 are equivalent.*

b) *If, in addition, Z and φ are as in Theorem 1. b), and $\{T_n\}_{n=1}^{\infty}$ satisfies Jackson and Bernstein-type inequalities of order $\beta \geqq 0$ on X with respect to Z, then assertions* (i)—(iv) *of Theorem 1 are equivalent.*

This result in the more general setting of intermediate spaces but only for $\varphi(t) = t^{\sigma}$, $\sigma > 0$ can be found in [7], [8], [9], [10].

4.2. Weak* approximation processes. The LAP $\{T_n\}$ of Theorem 2 maps X into the subspace Y. Hence the dual operators T_n' defined by

$$\langle T_n' f', f\rangle := \langle f', T_n f\rangle \quad (f' \in Y';\ f \in X) \tag{4.2}$$

map Y' into X'. In order to have $\{T_n'\}$ as an $\mathcal{M}$-LAP on Y' for a suitable $\mathcal{M}$ in Y'', we need to make some further assumptions. It will turn out that in this case the Jackson and Bernstein-type inequalities posed upon $\{T_n\}$ imply those upon $\{T_n'\}$.

Lemma 8. *Let X, Y be Banach spaces, $Y \subset X$* [3]*, and $\{T_n\}_{n=1}^{\infty}$ a* LAP *on*

[3]) "$Y \subset X$" means that Y is a continuously embedded subspace of X.

X satisfying Jackson and Bernstein-type inequalities of order $\alpha \geqq 0$ on X with respect to Y. Furthermore, assume that

$$\lim_{n\to\infty} \|T_n g - g\|_Y = 0 \quad (g \in Y). \tag{4.3}$$

Then $X' \subset Y'$ and $\{T_n'\}_{n=1}^{\infty}$ is a $J(Y)$-LAP on Y' (J being the canonical map from Y into Y'') that satisfies Jackson and Bernstein-type inequalities of order α on Y' with respect to X'.

Proof. Since Y is continuously embedded and dense in X, (density holding in view of the Bernstein-type inequality and (4.1)), it follows that $X' \subset Y'$. Now one has for $f' \in Y', f \in Y$ that

$$|\langle T_n' f' - f', f\rangle| = |\langle f', T_n f - f\rangle| \leqq \|f'\|_{Y'} \|T_n f - f\|_Y = o(1) \quad (n \to \infty)$$

in view of (4.3). This shows that $T_n' f'$ converges in the w^*-sense to f' or equivalently, that $J(Y) - \lim_{n\to\infty} V_n' f' = f'$. Furthermore, the commutativity of the sequence $\{T_n\}$ implies that of $\{T_n'\}$, and it follows that $\{T_n'\}$ is a $J(Y)$-LAP on Y'. Note that $J(Y)$ is closed in Y'', since Y is a Banach space. Finally, if I denotes the identity operator in any space, the Jackson and Bernstein-type inequalities foı $\{T_n'\}$ on Y' with respect to X' follow from (3.2) and (3.3) since (cf. [19, p. 214])

$$\|T_n' - I\|_{[X', Y']} = \|T_n - I\|_{[Y, X]} \leqq M n^{-\alpha} \quad (n \in \mathbf{N}),$$

$$\|T_n'\|_{[Y', X']} = \|T_n\|_{[X, Y]} \leqq M n^{\alpha} \quad (n \in \mathbf{N}).$$

Since for a Banach space $Y, J(Y)$-convergence is the same as w^*-convergence, in the following we will speak of a w^*-LAP instead of a $J(Y)$-LAP.

Lemma 8 enables one to apply Theorem 1. a) to the sequence $\{T_n'\}$. In order to obtain a counterpart also of Theorem 1. b) we take a Banach space Z such that $Y \subset Z \subset X$, and assume that $\{T_n\}$ satisfies Jackson and Bernstein-type inequalities of order $\beta \in [0, \alpha)$ on X with respect to Z. Then $X' \subset Z' \subset Y'$ by Lemma 8, and it follows from Lemma 7 and Lemma 8 that there holds a Bernstein-type inequality of order $\alpha - \beta$ on Y' with respect to Z' for $\{T_n'\}$. Concerning the Jackson-type inequality one has

Lemma 9. *Let $X, Y, \{T_n\}_{n=1}^{\infty}$ be given as in Lemma 8, and assume that there is a Banach space Z such that $Y \subset Z \subset X$ and $\{T_n\}$ satisfies Jackson and Bernstein-type inequalities of order $\beta \in [0, \alpha)$ on X with respect to Z. Then $\{T_n\}$ and $\{T_n'\}$ satisfy Jackson-type inequalities of order $\alpha - \beta$ on Z with respect to Y and on Y' with respect to Z', respectively.*

Proof. Since $Y \subset Z$, one has by (4.3) for $f \in Y$ that

$$\|T_n f - f\|_Z = \left\| \sum_{j=0}^{\infty} T_{n2^j} f - T_{n2^{j+1}} f \right\|_Z \leqq$$
$$\leqq \sum_{j=0}^{\infty} \{\|T_{n2^j}(f - T_{n2^{j+1}} f)\|_Z + \|T_{n2^{j+1}}(f - T_{n2^j} f)\|_Z\}.$$

Estimating the right side first by means of the Bernstein-type inequality on X with respect to Z, and then by the Jackson-type inequality on X with respect to Y, yields

$$\|T_n f - f\|_Z \leqq M \sum_{j=0}^{\infty} \{(n2^j)^{\beta}(n2^{j+1})^{-\alpha} + (n2^{j+1})^{\beta}(n2^j)^{-\alpha}\} \|f\|_Y \leqq$$
$$\leqq M n^{-(\alpha-\beta)} \|f\|_Y \quad (n \in \mathbf{N}).$$

This is the first part of the assertion; the second follows by Lemma 8.

We are now in a position to apply Theorem 1 to the dual of a LAP.

Theorem 3. a) *Let* X, Y *be Banach spaces such that* $Y \subset X$, *and* $\{T_n\}_{n=1}^{\infty}$ *a* LAP *on* X *satisfying* (4.3) *as well as Jackson and Bernstein-type inequalities of order* $\alpha > 0$ *on* X *with respect to* Y *and let* $\varphi \in \Phi$ *be such that* (3.5) *holds. Then* $X' \subset Y'$, $\{T_n'\}$ *defined by* (4.2) *is a* w^*-LAP *on* Y', *and the following assertions are equivalent for* $f' \in Y'$:

(i) $\|T_n' f' - f'\|_{Y'} = O(\varphi(n^{-1})) \quad (n \to \infty)$,

(ii) $K(t^{\alpha}, f'; Y', X') = O(\varphi(t)) \quad (t \to 0+)$.

b) *Suppose that* Z *is another Banach space such that* $Y \subset Z \subset X$, *and* $\{T_n\}$ *satisfies Jackson and Bernstein-type inequalities of order* $\beta \in [0, \alpha)$ *on* X *with respect to* Z, *and assume, that* (3.6) *holds for the order* $\alpha - \beta$ *instead of for* β. *Then each of the following assertions is equivalent to those of part* a) *for* $f' \in Y'$:

(iii) $f' \in Z'$ *and* $\|T_n' f' - f'\|_{Z'} = O(n^{\alpha-\beta} \varphi(n^{-1})) \quad (n \to \infty)$,

(iv) $\|T_n' f'\|_{X'} = O(n^{\alpha} \varphi(n^{-1})) \quad (n \to \infty)$.

The equivalence of assertions (i), (ii) and (iii) is contained in [17], [18] where it was proved by methods parallel to Theorem 2 in the frame of intermediate spaces.

If the commutativity (2.4) or even (3.13) is dropped, the method of obtaining equivalence theorems by using a strong Bernstein-type inequality in connection with the best approximation (3.14) cannot be carried over to the dual case. This is due to the fact that when passing to best approximation in dual spaces the strong Bernstein inequality converts into a Jackson inequality (for dual best approximation) and, conversely, the Jackson inequality for best approximation converts into a Bernstein inequality. This is entirely different from the situation for $\mathcal{M}$-LAP as was shown in Lemma 8. (For best approximation in dual spaces see [9], [17].)

On the other hand the approach using the stability condition (3.15) can always be applied to the dual case, since the dual stability condition needed, namely $\|T_n' g'\|_{X'} \leqq M \|g'\|_{X'}$ for all $g' \in X'$, $n \in \mathbf{N}$, holds for every LAP $\{T_n\}$ in view of (2.6).

5. Applications to convolution integrals

5.1. Results for the space $C_{2\pi}$. In this section we consider LAP's generated by convolution integrals. Let $C_{2\pi}$ denote the space of all continuous, 2π-periodic, complex-valued functions f defined on the real axis $\mathbf{R}$, endowed with the supremum norm $\|f\|_\infty$. A sequence of functions $\{\chi_n\}_{n=1}^\infty$ in $C_{2\pi}$ is called an approximate identity, if $\int_{-\infty}^{\infty} \chi_n(u)\,du = 2\pi$ for all $n \in \mathbf{N}$, and $\lim_{n\to\infty} \int_{|u|\geqq\delta} |\chi_n(u)|\,du = 0$ for each $\delta > 0$. The convolution integrals of $f \in C_{2\pi}$ with χ_n are defined by

$$(5.1) \qquad (V_n f)(x) \equiv (f * \chi_n)(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u)\chi_n(u)\,du \quad (n \in \mathbf{N};\ x \in \mathbf{R}).$$

The V_n are bounded linear operators from $C_{2\pi}$ into itself satisfying

$$(5.2) \qquad V_n V_m f = V_m V_n f, \quad \lim_{n\to\infty} \|V_n f - f\|_\infty = 0 \quad (f \in C_{2\pi};\ m, n \in \mathbf{N}).$$

Hence $\{V_n\}$ is a LAP on $C_{2\pi}$.

As subspaces Y and Z we take the Banach spaces $C_{2\pi}^k := \{f \in C_{2\pi};\ f^{(k)} \in C_{2\pi}\}$[4] for different values of $k \in \mathbf{P}$, endoved with the norm

$$\|f\|_{\infty,k} := |f^\wedge(0)| + \|f^{(k)}\|_\infty,$$ [5]

$f^\wedge(j)$ being the jth Fourier coefficient of f, namely

$$f^\wedge(j) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) e^{-iju}\,du \quad (j \in \mathbf{Z} := \{0, \pm 1, \pm 2, \dots\}).$$

Note that $V_n f \in C_{2\pi}^k$ for each $f \in C_{2\pi}$ provided $\chi_n \in C_{2\pi}^k$.

Concerning the function $\varphi \in \Phi$ we restrict ourselves to the case $\varphi(t) = t^\sigma$ for some $\sigma > 0$. Conditions (3.5) and (3.6) then read $\alpha > \sigma$ and $\beta < \sigma$, respectively.

---

[4]) Derivatives are denote by $f^{(1)}, f^{(2)}, \dots$, whereas the prime in f', g' indicates that these are elements of a dual space.

[5]) One may also take any equivalent norm, e.g. $\|f\|_\infty + \|f^{(k)}\|_\infty$, in particular $\|f\|_\infty$ instead of $\|f\|_{\infty,0}$.

Theorem 4. *Let* $\{\chi_n\}_{n=1}^{\infty}$ *be an approximate identity on* $C_{2\pi}$ *such that the associated convolution integrals* $\{V_n\}_{n=1}^{\infty}$ *(cf.* (5.1)*) satisfy Jackson and Bernstein-type inequalities of order* $\alpha>0$ *on* $C_{2\pi}$ *with respect to* $C_{2\pi}^r$ *for some (fixed)* $r\in\mathbf{N}$. *The following assertions are equivalent for* $f\in C_{2\pi}$, $s\in\mathbf{P}$, $\sigma>0$ *with* $\sigma s/r<\sigma<\alpha$:

(i) $\|V_nf-f\|_\infty = O(n^{-\sigma}) \quad (n\to\infty)$,

(ii) $K(t^\alpha, f;\ C_{2\pi}, C_{2\pi}^r) = O(t^\sigma) \quad (t\to 0+)$,

(iii) $f\in C_{2\pi}^s$ *and* $\|V_nf-f\|_{\infty,s} = O(n^{-\sigma+\alpha s/r}) \quad (n\to\infty)$,

(iv) $\|V_nf\|_{\infty,r} = O(n^{\alpha-\sigma}) \quad (n\to\infty)$.

Proof. The equivalence of (i) and (ii) follows immediately by Theorem 2. a). To prove that of (iii) as well as (iv) with (i) or (ii) we have to show that the assumptions of Theorem 2. b) are satisfied. It suffices to verify Jackson and Bernstein-type inequalities of order $\beta=\alpha s/r$ on $C_{2\pi}$ with respect to $Z=C_{2\pi}^s$. This is achieved by

Lemma 10. *Let* $\{V_n\}$ *be given as above,* $r\in\mathbf{N}$, $s\in\mathbf{P}$ *with* $s<r$. *If* $\{V_n\}$ *satisfies a Jackson-type inequality of order* $\alpha>0$ *on* $C_{2\pi}$ *with respect to* $C_{2\pi}^r$, *then there holds a Jackson-type inequality of order* $\alpha s/r$ *on* $C_{2\pi}$ *with respect to* $C_{2\pi}^s$. *The same holds for the Bernstein-type inequality.*

Proof. Consider the integral operator I^ν defined for $\nu\in\mathbf{P}$ on $C_{2\pi}$ via the Fourier series

$$(I^\nu f)(x)\sim f^\wedge(0)+\sum_{j=-\infty}^{\infty}{}' (ij)^{-\nu}f^\wedge(j)e^{-ijx}. \tag{5.3}$$

[6]

It is a linear operator from $C_{2\pi}^\mu$ into $C_{2\pi}^{\mu+\nu}$, $\mu,\nu\in\mathbf{P}$, having the properties

$$V_nI^\nu f = I^\nu V_nf \quad (f\in C_{2\pi};\ n\in\mathbf{N};\ \nu\in\mathbf{P}), \tag{5.4}$$

$$(I^\nu f)^{(\mu)} = f^{(\mu-\nu)} \quad (f\in C_{2\pi}^{\mu-\nu};\ \mu,\nu\in\mathbf{N}, \mu\geqq\nu). \tag{5.5}$$

Furthermore, we need Landau's inequality, namely that

$$\|f^{(\nu)}\|_\infty^\mu \leqq M_{\mu,\nu}\|f\|_\infty^{\mu-\nu}\|f^{(\mu)}\|_\infty^\nu \quad (f\in C_{2\pi}^\mu) \tag{5.6}$$

is valid for $\mu,\nu\in\mathbf{N}$ with $\nu\leqq\mu$ (cf. [16, p. 138]).

Now one has for $f\in C_{2\pi}^s$

$$\|V_nf-f\|_\infty^r = \|(V_nI^{r-s}f-I^{r-s}f)^{(r-s)}\|_\infty^r \leqq$$
$$\leqq M\|V_nI^{r-s}f-I^{r-s}f\|_\infty^s\,\|(V_nI^{r-s}f-I^{r-s}f)^{(r)}\|_\infty^{r-s}.$$

Estimating the first factor by the Jackson-type inequality on $C_{2\pi}$ with respect to $C_{2\pi}^r$ and using the equality $(V_nI^{r-s}f)^{(r)}=V_n((I^{r-s}f)^{(r)})$ together with (2.6) for the

[6]) The prime at Σ' indicates that the term for $j=0$ is to be omitted.

second, yields

$$\|V_n f - f\|_\infty^r \leqq M\{n^{-\alpha}\{|f^\wedge(0)| + \|(I^{r-s}f)^{(r)}\|_\infty\}^s \|(I^{r-s}f)^{(r)}\|_\infty^{r-s}.$$

The desired inequality now follows by (5.5).

To prove the second part of the lemma we use (5.6), (2.6) and the Bernstein-type inequality with respect to $C_{2\pi}^r$ to deduce

$$\|(V_n f)^{(s)}\|_\infty^r \leqq M\|V_n f\|_\infty^{r-s}\|(V_n f)^{(r)}\|_\infty^s \leqq M\|f\|_\infty^{r-s} n^{\alpha s}\|f\|_\infty^s.$$

The rest is obvious.

Remark. In view of the equivalence of the K-functional with the modulus of continuity (cf. [4]) one can express assertion (ii) of Theorem 4 in terms of Lipschitz spaces. Moreover, using the concept of fractional order derivatives (cf. [11]) the assumptions $r \in \mathbf{N}$, $s \in \mathbf{P}$ can be weakend to $r > 0$ and $s \geqq 0$. Note that all results remain valid when $C_{2\pi}$ is replaced by $L_{2\pi}^p$, $1 \leqq p < \infty$.

As particular examples of approximate identities $\{\chi_n\}$ one may take, e.g., the kernels $\{j_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ of Jackson and de la Vallée Poussin, namely

$$j_n(x) := \frac{3}{n(2n^2+1)}\left[\frac{\sin(nx/2)}{\sin(x/2)}\right]^4 \quad (n \in \mathbf{N};\ x \in \mathbf{R}),$$

$$v_n(x) := \frac{(n!)^2}{(2n)!}(2\cos(x/2))^{2n} \quad (n \in \mathbf{N};\ x \in \mathbf{R}).$$

For $\{j_n\}$ there hold Jackson and Bernstein-type inequalities of order 2 with respect to $C_{2\pi}^2$ (see [5], [6, p. 100]). Hence Theorem 4 holds for $\{f * j_n\}_{n=1}^\infty$ with $\alpha = r = 2$ and $s < \sigma < 2$. In case of $\{v_n\}$ Jackson and Bernstein-type inequalities of order 1 with respect to $C_{2\pi}^2$ are available ([5], [6, p. 113]). So Theorem 4 is valid for $\{f * v_n\}_{n=1}^\infty$ with $\alpha = 1$, $r = 2$ and $s/2 < \sigma < 1$.

5.2. Dual results. If $\{\chi_n\}_{n=1}^\infty$ is an approximate identity on $C_{2\pi}$, then one has in addition to (5.2) that

$$\lim_{n\to\infty} \|(V_n f)^{(k)} - f^{(k)}\|_\infty = \lim_{n\to\infty} \|V_n f - f\|_{\infty,k} = 0 \quad (f \in C_{2\pi}^k),$$

i.e., $\{V_n\}$ satisfies the assumptions of Theorem 3, too. Before applying this theorem to the $\{V_n\}$ we compute the duals of $C_{2\pi}^k$ and V_n.

To this end we regard $C_{2\pi}$ and $(C_{2\pi}^k)'$ as subspaces of $\mathscr{D}_{2\pi}'$, the space of all 2π-periodic distributions (cf. [20, Chap. 11]). On $\mathscr{D}_{2\pi}'$ we consider the operators $(k \in \mathbf{Z})$

$$(W^k f')(x) := \sum_{j=-\infty}^{\infty}{}' (ij)^k f'^\wedge(j) e^{ijx} \quad (f' \in \mathscr{D}_{2\pi}'),$$

$$W_0^k f' := W^k f' + f'^\wedge(0) \quad (f' \in \mathscr{D}_{2\pi}'),$$

where the convergence is to be understood in the topology of $\mathcal{D}'_{2\pi}$, and the distributional Fourier coefficients are given by $f'^{\wedge}(j):=(2\pi)^{-1}\langle f'(x), e^{-ijx}\rangle$. For $k\in\mathbf{N}$ the operator W^k coincides on $C^k_{2\pi}$ with the usual kth order derivative, and W_0^{-k} is the same as I^k on $C_{2\pi}$.

Now we extend the definition of $C^k_{2\pi}$, $k\in\mathbf{P}$, to arbitrary $k\in\mathbf{Z}$ by

$$C^k_{2\pi}:=\{f'\in\mathcal{D}'_{2\pi};\ W^k(f')\in C_{2\pi}\},\quad \|f'\|_{\infty,k}:=|f'^{\wedge}(0)|+\|W^k(f')\|_\infty,$$

and furthermore set

$$(C'_{2\pi})^k:=\{f'\in\mathcal{D}'_{2\pi};\ W^kf'\in C'_{2\pi}\},\quad \|f'\|_{(C'_{2\pi})^k}:=|f'^{\wedge}(0)|+\|W^kf'\|_{C'_{2\pi}}.$$

The norm for $(C'_{2\pi})^0=C'_{2\pi}$ introduced here is equivalent to that which is usually used.

Since W_0^{-k} is a linear homeomorphism from $C_{2\pi}$ onto $C^k_{2\pi}$, it follows that the dual operator $(W_0^{-k})'$ defined by

$$\langle (W_0^{-k})'f', f\rangle := \langle f', W_0^{-k}f\rangle\quad (f'\in\mathcal{D}'_{2\pi};\ f\in C^\infty_{2\pi})$$

is a linear homeomorphism from $(C^k_{2\pi})'$ onto $C'_{2\pi}$. Hence one can rewrite

$$(C^k_{2\pi})'=\{f'\in\mathcal{D}'_{2\pi};\ W_0^{-k}f'\in C'_{2\pi}\}.$$

Comparing this with the definition of $(C'_{2\pi})^{-k}$, and noting that $(W_0^{-k})'f'= =f'^{\wedge}(0)+(-1)^kW^{-k}f'$, one has that $(C^k_{2\pi})'=(C'_{2\pi})^{-k}$ in the set theoretical sense. Concerning the norms, there holds by the properties of $(W_0^{-k})'$

$$\|f'\|_{(C^k_{2\pi})'}\le M\|(W_0^{-k})'f'\|_{C'_{2\pi}}\le M\{|f'^{\wedge}(0)|+\|W^{-k}f'\|_{C'_{2\pi}}\}=M\|f'\|_{(C'_{2\pi})^{-k}}.$$

Since the converse inequality follows by the same arguments, we have proved (cf. [17])

Lemma 11. *The spaces $(C^k_{2\pi})'$ and $(C'_{2\pi})^{-k}$ are equal with equivalent norms, in notation $(C^k_{2\pi})'\cong(C'_{2\pi})^{-k}$.*

Note that this result is also valid for arbitrary $k\in\mathbf{R}$, working in the fractional frame.

Next we compute the dual operators V'_n assuming for simplicity $\{\chi_n\}_{n=1}^\infty$ to be an even approximate identity, i.e., $\chi_n(u)=\chi_n(-u)$ for all $n\in\mathbf{N}$, $u\in\mathbf{R}$. First we extend the domain of V_n from $C_{2\pi}$ to $\mathcal{D}'_{2\pi}$ via

$$(V_nf')(x)=\sum_{j=-\infty}^{\infty} f'^{\wedge}(j)\chi_n^{\wedge}(j)e^{ijx}\quad (f'\in\mathcal{D}'_{2\pi};\ n\in\mathbf{N})$$

which obviously coincides for $f'\in C_{2\pi}$ with (5.1). For the dual V_n' one has in view of $\chi_n^\wedge(j)=\chi_n^\wedge(-j)$ that

$$\langle V_n'f',f\rangle = \langle f',V_nf\rangle = \sum_{j=-\infty}^{\infty} f'^\wedge(-j)\chi_n^\wedge(j)f^\wedge(j) = \langle V_nf',f\rangle \quad (f\in C_{2\pi}^\infty),$$

hence $V_n'=V_n$. (If one drops the assumption χ_n to be even, then one would have $V_n'=V_n^-$, where $(V_n^-f)(x):=(f(\cdot)*\chi_n(-\cdot))(x)$.) Observing these various facts, we can formulate as a consequence of Theorem 3

Theorem 5. *Let $\{\chi_n\}_{n=1}^\infty$ be an even approximate identity on $C_{2\pi}$ such that the associated convolution integrals $\{V_n\}_{n=1}^\infty$ satisfy Jackson and Bernstein-type inequalities of order $\alpha>0$ on $C_{2\pi}$ with respect to $C_{2\pi}^r$ for some $r\in\mathbf{N}$. The following assertions are equivalent for $f'\in(C_{2\pi}^r)'\cong(C_{2\pi}')^{-r}$, $s\in\mathbf{P}$, $\sigma>0$ with $0<\alpha-\alpha s/r<\alpha$:*

(i) $\|V_nf'-f'\|_{(C_{2\pi}')^{-r}} = O(n^{-\sigma}) \quad (n\to\infty)$,

(ii) $K(t^\alpha,f';\,(C_{2\pi}')^{-r},C_{2\pi}') = O(t^\sigma) \quad (t\to 0+)$,

(iii) $f'\in(C_{2\pi}^s)'\cong(C_{2\pi}')^{-s}$ *and* $\|V_nf'-f'\|_{(C_{2\pi}')^{-s}} = O(n^{-\sigma+\alpha-\alpha s/r}) \quad (n\to\infty)$,

(iv) $\|V_nf'\|_{C_{2\pi}'} = O(n^{-\sigma+\alpha}) \quad (n\to\infty)$.

If one sets $g':=I^rf'$, then $g'\in C_{2\pi}'$, and each of the following is equivalent to the corresponding assertion for f':

(i)′ $\|V_ng'-g'\|_{C_{2\pi}'} = O(n^{-\sigma}) \quad (n\to\infty)$,

(ii)′ $K(t^\alpha;\,g';\,C_{2\pi}',(C_{2\pi}')^r) = O(t^\sigma) \quad (t\to 0+)$,

(iii)′ $g'\in(C^{s-r})'\cong(C_{2\pi}')^{r-s}$ and

$\|V_n(W^{r-s}g')-W^{r-s}g'\|_{C_{2\pi}'} = O(n^{-\sigma-\alpha-\alpha s/r}) \quad (n\to\infty)$,

(iv)′ $\|V_n(W^rg')\|_{C_{2\pi}'} = O(n^{-\sigma+\alpha}) \quad (n\to\infty)$.

Using the fact that $C_{2\pi}'$ is isometrically isomorphic to $BV[-\pi,\pi]$, the space of all complex-valued functions of bounded variation on $[-\pi,\pi]$, one can rewrite statements (i)′—(iv)′ in terms of functions $\mu\in BV[-\pi,\pi]$. Note that if $g'\in C_{2\pi}'$ corresponds to $\mu\in BV[-\pi,\pi]$, then V_ng' corresponds to $(\tilde\chi_n*d\mu)(x):=$ $:=(2\pi)^{-1}\int_{-\pi}^{\pi}\tilde\chi_n(x-u)\,d\mu(u)$, where $\tilde\chi_n(u):=\int_{-\pi}^{u}\chi_n(u)\,du$. We omit the details.

It would be of interest to find an approximation process that satisfies the assumptions of Theorem 3 but differs from its dual operator.

6. Appendix

The aim of this section is to express conditions (3.5) and (3.6) upon $\varphi\in\Phi$ by equivalent ones, which can be verified more easily. Although these results are implicitly contained in [2] we present the proofs for completeness.

Lemma 12. *Let* $\varphi\in\Phi$, $\alpha,\beta\geqq 0$. a) *If*

$$t^{-\alpha}\varphi(t)\leqq Ms^{-\alpha}\varphi(s)\quad(0<s\leqq t\leqq 1),\tag{6.1}$$

then assertion (3.5) *is equivalent to*

$$\overline{\lim_{t\to 0+}}\frac{\varphi(Ct)}{\varphi(t)}<C^{\alpha}\tag{6.2}$$

for some $C>1$.

b) *Under the assumption*

$$s^{-\beta}\varphi(s)\leqq Mt^{-\beta}\varphi(t)\quad(0<s\leqq t\leqq 1)\tag{6.3}$$

assertion (3.6) *is equivalent, for some* $C>1$, *to*

$$\underline{\lim}_{t\to 0+}\frac{\varphi(Ct)}{\varphi(t)}>C^{\beta}.\tag{6.4}$$

Proof. a) It follows from (3.5) that there exists a constant $M>0$ such that

$$\sum_{2^{-r}t^{-1}\leqq 2^j\leqq t^{-1}}2^{\alpha j}\varphi(2^{-j})\leqq Mt^{-\alpha}\varphi(t)\quad(r\in\mathbf{N};\ 0\leqq t\leqq 2^{-r}).$$

Since the sum consists of at least r terms, each of which is $\geqq M2^{-\alpha r}t^{-\alpha}\varphi(2^r t)$ by (6.1) one obtains

$$r2^{-\alpha r}t^{-\alpha}\varphi(2^r t)\leqq Mt^{-\alpha}\varphi(t)\quad(r\in\mathbf{N};\ 0\leqq t\leqq 2^{-r}).\tag{6.5}$$

Choosing now $r\in\mathbf{N}$ greater than the constant M in (6.5) yields (6.2) with $C=2^r$.

Conversely, let $k\in\mathbf{N}$ be such that $2^{k-1}<C\leqq 2^k$. Then, in view of (6.1) and (6.2), one can find q, $t_0\in(0,1)$ such that

$$\frac{\varphi(t)}{2^{k\alpha}\varphi(t2^{-k})}\leqq q\quad\bigl(t\in(0,t_0)\bigr).\tag{6.6}$$

Now one has for $m_0, m\in\mathbf{N}$ satisfying $2^{k(m_0-1)}\leqq t_0^{-1}<2^{km_0}$, $2^{k(m-1)}\leqq t^{-1}<2^{km}$ that

$$\sum_{1\leqq 2^j\leqq t^{-1}}2^{\alpha j}\varphi(2^{-j})\leqq\Bigl(\sum_{j=0}^{km_0-1}+\sum_{j=km_0}^{km}\Bigr)2^{\alpha j}\varphi(2^{-j})\leqq$$

$$\leqq M+\sum_{\nu=m_0}^{m}\sum_{j=k\nu}^{k(\nu+1)-1}2^{\alpha j}\varphi(2^{-j})\leqq M+\sum_{\nu=m_0}^{m}\varphi(2^{-k\nu})\sum_{j=k\nu}^{k(\nu+1)-1}2^{\alpha j}=$$

$$=M+\sum_{\nu=m_0}^{m}\varphi(2^{-k\nu})2^{\alpha k\nu}\frac{2^{\alpha k}-1}{2^{\alpha}-1}\leqq M\{1+\sum_{\nu=m_0}^{m}2^{\alpha k\nu}\varphi(2^{-k\nu})\}\quad\bigl(t\in(0,t_0)\bigr).$$

Since $2^{-kv}<t_0$ for $v \geqq m_0$, one can estimate the latter sum by (6.6) to deduce

$$\sum_{v=m_0}^{m} 2^{\alpha kv}\varphi(2^{-kv}) \leqq 2^{\alpha km}\varphi(2^{-km}) \sum_{v=m_0}^{m} \frac{\varphi(2^{-kv})}{2^{(m-v)}\varphi(2^{-(m-v)k})} \leqq$$

$$\leqq 2^{\alpha km} \sum_{v=m_0}^{m} q^{m-v} \leqq M 2^{\alpha km}.$$

This yields, in view of (6.1),

$$\sum_{1 \leqq 2^j \leqq t^{-1}} 2^{\alpha j}\varphi(2^{-j}) \leqq M\{1+2^{\alpha km}\varphi(2^{-km})\} \leqq Mt^{-\alpha}\varphi(t) \quad \big(t\in(0, t_0)\big)$$

which is assertion (3.5).

Concerning part b) set $\psi(t):=t^{-\beta}\varphi(t)$. Then, replacing t by $2^r t$ in (3.6), one has

$$\sum_{j=0}^{\infty} \psi(t2^{r-j}) \leqq M\psi(t2^r) \quad (r\in\mathrm{N};\ 0<t\leqq 2^{-r}).$$

By (6.3) it follows that $\psi(t2^{r-j}) \leqq M\psi(t)$ for $j=0, 1, \dots, r$, and so

$$(r+1)\psi(t) \leqq M\psi(t2^r) \quad (r\in\mathrm{N};\ 0<t\leqq 2^{-r}).$$

This in turn implies (6.4) by choosing $r \geqq M$ and $C=2^r$.

Conversely, if (6.4) holds and $k\in\mathrm{N}$ satisfies $2^{k-1}<C\leqq 2^k$, then

$$\frac{\psi(t2^{-k})}{\psi(t)} \leqq q < 1 \quad \big(t\in(0, t_0)\big) \tag{6.7}$$

for suitable $q, t_0\in(0, 1)$. Then one obtains by (6.3) and (6.7) that

$$\sum_{j=0}^{\infty} \psi(t2^{-j}) = \sum_{v=0}^{\infty} \sum_{j=kv}^{k(v+1)-1} \psi(t2^{-j}) \leqq M \sum_{v=0}^{\infty} k\psi(t2^{-kv}) \leqq$$

$$\leqq M\psi(t) \sum_{v=0}^{\infty} \frac{\psi(t2^{-kv})}{\psi(t)} \leqq M\psi(t) \sum_{v=0}^{\infty} q^v \quad (0<t\leqq t_0).$$

This gives (3.6), and so our proof is complete.

It should be mentioned that condition (6.1) is superfluous. Indeed, it can be shown that it follows from (3.5) as well as from (6.2). The proofs would then become more intricate. In this respect see [2], where also some further equivalent assertions to those of part a) and b) can be found.

Using Lemma 12 one can easily show that $\varphi(t):=(\log 1/t)^{-\gamma}$, $\gamma>0$, satisfies (3.5) for each $\alpha>0$, but that (3.6) is violated for each $\beta \geqq 0$. On the other hand, functions which behave like $e^{-1/t}$ are not admissible in our theorems since $\lim_{t\to 0+} t^{-\alpha}e^{-1/t}=0$ for each $\alpha\geqq 0$ which implies by Lemma 3 that (3.5) cannot hold. So one can handle the case where $T_n f$ converges very slowly towards f, but not the case where it converges very rapidly.

Acknowledgements. The authors would like to thank Professor Dr. P. L. Butzer, Aachen, for valuable comments and critical reading of the manuscript.

References

[1] S. BANACH, *Théorie des opérations linéaires*, Chelsea Publishing Co. (New York, 1955).
[2] N. K. BARI and S. B. STEČKIN, Best approximations and differential properties of two conjugate functions, *Trudy Moskov. Mat. Obšč.*, **5** (1956), 483—522.
[3] M. BECKER and R. J. NESSEL, Inverse results via smoothing, in: *Constructive Theory of Functions* (Proc. Internat. Conf. Blagoevgrad, 1977; Eds. Bl. Sendov and D. Vacov), Publishing House Bulg. Acad. Sci. (Sofia, 1980); pp. 231—243.
[4] P. L. BUTZER, H. DYCKHOFF, E. GÖRLICH and R. L. STENS, Best trigonometric approximation, fractional order derivatives and Lipschitz classes, *Canad. J. Math.*, **29** (1977), 781—793.
[5] P. L. BUTZER and J. JUNGGEBURTH, On Jackson-type inequalities in approximation theory, in: *General Inequalities 1* (Proc. Internat. Conf. Oberwolfach, 1976; Ed. E. F. Beckenbach), ISNM, Vol. 41, Birkhäuser Verlag (Basel, 1978); pp. 85—114.
[6] P. L. BUTZER and R. J. NESSEL, *Fourier Analysis and Approximation*, Vol. 1, Birkhäuser Verlag (Basel); Academic Press (New York, 1971).
[7] P. L. BUTZER and K. SCHERER, Über die Fundamentalsätze der klassischen Approximationstheorie in abstrakten Räumen, in: *Abstract Spaces and Approximation* (Proc. Internat. Conf. Oberwolfach, 1968; Eds. P. L. Butzer and B. Sz.-Nagy), ISNM, Vol. 10, Birkhäuser Verlag (Basel, 1969); pp. 113—125.
[8] P. L. BUTZER and K. SCHERER, On the fundamental approximation theorems of D. Jackson, S. N. Bernstein and theorems of M. Zamansky and S. B. Stečkin, *Aequationes Math.*, **3** (1969), 170—195.
[9] P. L. BUTZER and K. SCHERER, On fundamental theorems of approximation theory and their dual versions, *J. Approx. Theory*, **3** (1970), 87—100.
[10] P. L. BUTZER and K. SCHERER, Approximation theorems for sequences of commutative operators in Banach spaces, in: *Constructive Theory of Functions* (Proc. Internat. Conf. Varna, 1970; Eds. B. Penkov and D. Vacov), Publishing House Bulg. Acad. Sci. (Sofia, 1972); pp. 137—145.
[11] P. L. BUTZER and U. WESTPHAL, An access to fractional differentiation via fractional difference quotients, in: *Fractional Calculus and Its Applications* (Proc. Internat. Conf. West Haven, Conn., 1974; Ed. B. Ross), Lecture Notes in Math., Vol. 457, Springer-Verlag, (Berlin, 1975); pp. 116—145.
[12] J. DIXMIER, Sur un théorème de Banach, *Duke Math. J.*, **15** (1948), 1057—1071.
[13] N. DUNFORD and J. T. SCHWARTZ, *Linear Operators*, Vol. 1, Interscience Publishers (New York—London, 1958).
[14] A. GRUNDMANN, Inverse theorems for Kantorovič polynomials. in: *Fourier Analysis and Approximation Theory* (Proc. Colloq. Budapest, 1976; Eds. G. Alexits and P. Turán), Vol. I, North-Holland Publishing Co. (Amsterdam, 1978); pp. 395—401.

[15] J. JUNGGEBURTH, K. SCHERER and W. TREBELS, Zur besten Approximation auf Banachräumen mit Anwendungen auf ganze Funktionen, in: *Forschungsberichte des Landes Nordrhein-Westfalen*, Nr. 2311, Westdeutscher Verlag (Opladen, 1973); pp. 51—75.

[16] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag (Berlin, 1970).

[17] K. SCHERER, *Dualität bei Interpolations- und Approximationsräumen*, Doctoral Dissertation, RWTH Aachen, 1969.

[18] K. SCHERER, Über die Dualen von Banachräumen, die durch lineare Approximationsprozesse erzeugt werden, und Anwendungen für periodische Distributionen, *Acta Math. Acad. Sci. Hungar.*, **23** (1972), 343—365.

[19] A. E. TAYLOR, *Introduction to Functional Analysis*, John Wiley & Sons (New York, 1958).

[20] A. H. ZEMANIAN, *Distribution Theory and Transform Analysis*, McGraw-Hill Book Co. (New York, 1965).

LEHRSTUHL A FÜR MATHEMATIK
RWTH AACHEN
TEMPLERGRABEN 55
5100 AACHEN, FED. REP. GERMANY

Acta Sci. Math., 47 (1984), 169—188

Steckin-type estimates for locally divisible multipliers in Banach spaces

W. DICKMEIS, R. J. NESSEL, and E. VAN WICKEREN[1])

1. Introduction

Let $X_{2\pi}$ be one of the Banach spaces $C_{2\pi}$ or $L^p_{2\pi}$, $1 \leqq p < \infty$, of 2π-periodic functions, continuous or Lebesgue measurable on the real axis $\mathbf{R}$ with finite norm

$$\|f\|_{C_{2\pi}} := \max_{u \in \mathbf{R}} |f(u)|, \quad \|f\|_{p,2\pi} := \left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(u)|^p \, du\right\}^{1/p},$$

respectively. Let $\mathbf{C}$ denote the complex plane and

$$(1.1) \qquad \Pi_n := \left\{p_n \in C_{2\pi};\ p_n(u) := \sum_{k=-n}^{n} c_k e^{iku},\ c_k \in \mathbf{C}\right\}$$

the set of trigonometric polynomials of degree at most $n \in \mathbf{N}$ (:=set of natural numbers). For $f \in X_{2\pi}$ the error of best approximation by elements of Π_n is denoted by

$$(1.2) \qquad E(f, n) := E(X_{2\pi};\ f, n) := \inf\{\|f - p_n\|_{X_{2\pi}};\ p_n \in \Pi_n\}.$$

Let the rth modulus of continuity of $f \in X_{2\pi}$ be given by ($r \in \mathbf{N}$)

$$(1.3) \qquad \omega_r(X_{2\pi};\ f, t) := \sup_{|h| \leqq t} \left\| \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(u + kh) \right\|_{X_{2\pi}}.$$

In these terms, STECKIN [15] proved in 1951 the following (weak-type, cf. [5]) inequality ($f \in X_{2\pi}$, $n \in \mathbf{N}$)

$$(1.4) \qquad \omega_r(X_{2\pi};\ f, n^{-1}) \leqq A_r n^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E(X_{2\pi};\ f, k)$$

Received November 2, 1982.

[1]) Supported by Deutsche Forschungsgemeinschaft Grant No. Ne 171/5—1.

which immediately furnishes the classical Bernstein inverse approximation theorem (cf. [19, p. 331 ff.]). Ten years later, STECKIN [16] considered the Fejér means

$$(F_{n-1}f)(u) := \sum_{k=-n}^{n} (1-|k|/n) f^{\wedge}(k) e^{iku}$$

of the Fourier series of $f \in X_{2\pi}$ where for $k \in \mathbf{Z}$ (:= set of integers) the kth Fourier coefficient is given by $2\pi f^{\wedge}(k) := \int_{-\pi}^{\pi} f(u) e^{-iku} du$. Corresponding to (1.4) he established the inequality $(f \in X_{2\pi}, n \in \mathbf{N})$

$$\|F_{n-1}f - f\|_{X_{2\pi}} \leqq A n^{-1} \sum_{k=0}^{n-1} E(X_{2\pi}; f, k). \tag{1.5}$$

In both cases, Steckin essentially used the same technique, namely Bernstein's classical telescope argument, employing the (unique) polynomial of best approximation in 1951 and the delayed means of de La Vallée Poussin in 1961, respectively. Moreover, in [16, p. 271] he pointed out that it would be interesting to obtain estimates, analogous to (1.5), for other methods of summation of Fourier series.

It is the purpose of this paper to derive inequalities of type (1.4, 5) for quite a general class of processes within the abstract framework of Banach spaces, admissible with respect to some Riesz-bounded spectral measure (see also the general approach given in [1a]).

To this end, multipliers are defined in Section 2 for Banach spaces which are generated via closure by some orthonormal structure given in terms of a spectral measure in a Hilbert space. If the spectral measure is Riesz-bounded, then a uniform bound can be derived for families of radial multipliers of Hardy-type (cf. Theorem 2.1). This enables one in Section 3 to introduce polynomials, potential spaces, and de La Vallée Poussin means, a basic tool. In fact, Sections 2 and 3 represent a brief outline of a general framework within which one may successfully treat a number of classical problems of approximation theory and numerical analysis (for details see [4], [12], [21] and the literature cited there).

To derive Steckin-type estimates, the concept of locally (globally) divisible multipliers is introduced in Section 4. Here we are heavily influenced by work of H. S. SHAPIRO [14] concerned with local divisibility within the Wiener ring of Fourier—Stieltjes transforms (see [14] and the literature cited there). In fact, whereas the present approach is finally based upon some Hilbert space structure (e.g., L^2), one may consult [1] for a different type of extension which deals with the local divisibility of Gelfand transforms in commutative Banach algebras (e.g., L^1). In any case, a first application of the present concepts leads to the Jackson-type inequality (4.2) and thus to the global Jackson-type theorem given in (4.3). The actual Steckin-type estimates are derived in Section 5. It is interesting to note that the Bernstein—

Steckin telescoping technique indeed extends to the present abstract situation. In this connection, let us mention that similar arguments in the setting of Besov spaces may also be found in [2], [10], 13], [23].

The sharpness of the classical estimates (1.4, 5) was already discussed by Steckin (cf. [16]) via concrete methods. But this kind of problems can also be dealt with in the present abstract setting. To this end, Section 6 first recalls a general theorem, established in [6], [7], which in fact does not need any orthogonal structure. Based upon some rather mild conditions upon the spectral measure and the locally divisible family of multipliers, Corollary 6.2 then shows that the assumptions of the theorem are indeed satisfied in the present context.

Finally, Section 7 is devoted to some first illustrating applications, emphasizing the unifying approach to the subject.

2. Multipliers

For a complex Hilbert space H let E be a (countably additive, selfadjoint, bounded, linear) spectral measure in $\mathbf{R}^N$, the Euclidean N-space ($N\in\mathbf{N}$) with inner product $xy:=\sum_{j=1}^{N} x_j y_j$ and norm $|x|:=(xx)^{1/2}$. Thus, E maps the family Σ of all Borel measurable sets in $\mathbf{R}^N$ into the set of all self-adjoint, bounded, linear projections of H such that ($B, B_j\in\Sigma$, $\emptyset$ being the void set, I the identity mapping)

$$
\text{(2.1)}\qquad
\begin{aligned}
&\text{(i)}\quad E(B_1\cap B_2) = E(B_1)E(B_2),\\
&\text{(ii)}\quad E(\emptyset) = 0,\quad E(\mathbf{R}^N) = I,\\
&\text{(iii)}\quad E\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} E(B_j)\quad (B_i\cap B_j = \emptyset \quad\text{for}\quad i\neq j).
\end{aligned}
$$

Let $L^\infty(\mathbf{R}^N, E)$ be the space of complex-valued, E-essentially bounded functions τ with norm

$$
\text{(2.2)}\qquad \|\tau\|_{\infty,E} := \inf\{\sup_{u\in B}|\tau(u)|;\ B\in\Sigma,\ E(B) = I\}.
$$

For each $\tau\in L^\infty(\mathbf{R}^N, E)$ the integral $T^\tau := \int_{\mathbf{R}^N}\tau(u)dE(u)$ is a bounded, linear operator of H into itself (for basic properties and further details see [8, pp. 900, 1930, 2186]).

For a given orthonormal structure (H, E) let X be a complex Banach space with norm $\|\cdot\|$ such that H and X are continuously embedded in some linear Hausdorff space (this hypothesis should be added in [4], see [23, p. 116]) and such that

$$
\text{(2.3)}\qquad \overline{H\cap X}^{\|\cdot\|_H} = H,\quad \overline{H\cap X}^{\|\cdot\|} = X,
$$

i.e., $H\cap X$ is dense in H and X. Then (cf. [4]) $\tau\in L^\infty(\mathbf{R}^N, E)$ is called a multiplier on X if for each $f\in H\cap X$

(2.4) $$T^\tau f:=\int_{\mathbf{R}^N}\tau(u)\,dE(u)f\in H\cap X,\quad \|T^\tau f\|\leqq A\|f\|.$$

In view of (2.3, 4), the closure of T^τ (represented by the same symbol) belongs to $[X]$, the space of bounded, linear operators of X into itself. The set of all multipliers τ on X is denoted by $M=M(X)=M(X,H,E)$, the corresponding set of multiplier operators T^τ by $[X]_M$. Setting

(2.5) $$\|\tau\|_M:=\|T^\tau\|_{[X]}:=\sup\{\|T^\tau f\|;\ f\in H\cap X,\ \|f\|\leqq 1\},$$

M is a commutative Banach algebra with unit under the natural vector operations and pointwise multiplication, isometrically isomorphic to the subspace $[X]_M\subset[X]$.

Let $D^{(j)}, j\in\mathbf{N}$, be the set of real-valued, continuous, strictly increasing functions ψ on $[0,\infty)$ with $\psi(0)=0$ and $\lim_{t\to\infty}\psi(t)=\infty$, which are $(j+1)$ times differentiable on $(0,\infty)$ with

(2.6) $$\text{(i)}\quad t^k|\psi^{(k+1)}(t)|\leqq K\psi'(t)\quad(0\leqq k\leqq j,\ t>0),$$
$$\text{(ii)}\quad \lim_{t\to0+}t\psi'(t)=0.$$

For $j=0$ set $D^{(0)}:=D^{(1)}$. In view of (2.6) one has

(2.7) $$t\psi'(t)\leqq\int_0^t[\psi'(u)+u|\psi''(u)|]\,du\leqq(K+1)\psi(t),$$
$$t^k\psi^{(k)}(t)\leqq Kt\psi'(t)\leqq K(K+1)\psi(t)\quad(0\leqq k\leqq j+1).$$

Thus ψ satisfies

(2.8) $$\psi(st)\leqq s^{K+1}\psi(t)\quad(s\geqq1,\ t\geqq0),$$

since (2.7) implies

$$\log\frac{\psi(st)}{\psi(t)}=\int_t^{st}\frac{\psi'(u)}{\psi(u)}\,du\leqq\int_t^{st}\frac{K+1}{u}\,du=\log s^{K+1}.$$

Note that $\psi(t)=t^\gamma, \gamma>0$, and $\psi(t)=\log(1+t)$ are admissible choices but not $\psi(t)=e^t$.

Let $\varphi(\varrho)$ be a real-valued, positive function on an index set $\mathcal{J}$. For $\psi\in D^{(j)}$ and a function σ, defined on $[0,\infty)$, the family $\{\sigma^\psi_{\varphi(\varrho)}\}_{\varrho\in\mathcal{J}}$ with $\sigma^\psi_{\varphi(\varrho)}(x):=$ $:=\sigma(\varphi(\varrho)\psi(|x|))$ is said to be of Hardy-type (φ,ψ) if $\sigma^\psi_{\varphi(\varrho)}$ belongs to M, uniformly for $\varrho\in\mathcal{J}$ (cf. [21] and the literature cited there).

To formulate a criterion for multipliers of this type, assume that for a Banach space X, satisfying (2.3), the spectral measure E is Riesz-(R,j)-bounded for some $j \in \mathbf{P}$ (:= set of non-negative integers), i.e.,

$$r_{j,\varrho}(x) := \begin{cases} (1-|x|/\varrho)^j & \text{for} \quad |x| \leqq \varrho \\ 0 & \text{for} \quad |x| > \varrho \end{cases}$$

belongs to M and $\|r_{j,\varrho}\|_M \leqq C_j$, uniformly for all $\varrho > 0$.

Theorem 2.1. *For some $j \in \mathbf{P}$ let E be a (R, j)-bounded spectral measure for X. Let $\psi \in D^{(j)}$ and let $\varphi(\varrho)$ be a positive function on $\mathcal{I}$. If $\sigma \in BV_{j+1}$, the class of (sufficiently smooth) functions satisfying*

$$\|\sigma\|_{BV_{j+1}} := \frac{1}{j!} \int_0^\infty t^j |d\sigma^{(j)}(t)| + \lim_{t\to\infty} |\sigma(t)| < \infty,$$

then the family $\{\sigma^{\psi}_{\varphi(\varrho)}\}_{\varrho \in \mathcal{I}}$ is of Hardy-type (φ, ψ). In fact, $\|\sigma^{\psi}_{\varphi(\varrho)}\|_M \leqq A \|\sigma\|_{BV_{j+1}}$ where A is independent of σ and $\varrho \in \mathcal{I}$.

Note that for $\psi(t) = t$, $\varphi(\varrho) = 1/\varrho$, and $\mathcal{I} = (0, \infty)$ Theorem 2.1 covers the multipliers of Fejér-type. For further details, including the fractional extension of the class BV_{j+1}, however, see [4], [21], [22] and the literature cited there.

3. Polynomials, de La Vallée Poussin means, potential spaces

For some $j \in \mathbf{P}$ let E be (R, j)-bounded for a Banach space X. Let $\{\mu_\varepsilon\}_{\varepsilon > 0}$ be a family of real-valued, infinitely differentiable functions on $[0, \infty)$ satisfying

$$0 \leqq \mu_\varepsilon(t) \leqq 1, \quad \mu_\varepsilon(t) = \begin{cases} 1, & 0 \leqq t \leqq 1 + \varepsilon/2, \\ 0, & t \geqq 1 + \varepsilon. \end{cases}$$

It follows from Theorem 2.1 that the family $\{T_{\varepsilon,\varrho}\}$,

$$T_{\varepsilon,\varrho} := T^{\mu_{\varepsilon,\varrho}}, \quad \mu_{\varepsilon,\varrho}(x) := \mu_\varepsilon(|x|/\varrho) \quad (\varepsilon, \varrho > 0),$$

is well-defined in $[X]_M$. The set of polynomials in X (of radial degree $\varrho > 0$) is then defined by (cf. [11], [12])

$$\Pi := \bigcup_{\varrho > 0} \Pi_\varrho, \quad \Pi_\varrho := \{f \in X;\ T_{\varepsilon,\varrho} f = f \quad \text{for all} \quad \varepsilon > 0\}.$$

In the following we shall call a Banach space X *admissible* (with respect to (H, E)) if X satisfies (2.3), E is (R, j)-bounded for some $j \in \mathbf{P}$, and if the polynomials are dense in X, i.e., $\overline{\Pi}^{\|\cdot\|} = X$. Obviously, the latter condition is equivalent to $(\varrho \to \infty)$

$$E(f, \varrho) := E(X;\ f, \varrho) := \inf\{\|p - f\|;\ p \in \Pi_\varrho\} = o(1),$$

where the error of best approximation $E(f, \varrho)$ (cf. (1.2)) is a decreasing function in $\varrho>0$.

Basic for the present treatment will be the family $\{L_\varrho\}_{\varrho>0}$ of de La Vallée Poussin (or delayed) means. For a real-valued, infinitely differentiable function λ on $[0, \infty)$ satisfying

$$(3.1) \qquad 0 \leqq \lambda(t) \leqq 1, \quad \lambda(t) = \begin{cases} 1, & 0 \leqq t \leqq 1, \\ 0, & t \geqq 2, \end{cases}$$

set $\lambda_\varrho(x):=\lambda(|x|/\varrho)$ and $L_\varrho:=T^{\lambda_\varrho}$. It follows from Theorem 2.1 that the operators L_ϱ are well-defined on each admissible space X. In fact, one has (cf. [11], [12])

Proposition 3.1. *Let X be an admissible Banach space. Then*

$$(3.2) \qquad \|L_\varrho\|_{[X]} \leqq A \quad (\varrho > 0),$$

$$(3.3) \qquad L_\varrho f \in \Pi_{2\varrho} \quad (f \in X, \varrho > 0),$$

$$(3.4) \qquad L_\varrho p = p \quad (p \in \Pi_\varrho, \varrho > 0),$$

$$(3.5) \qquad \|L_\varrho f - f\| \leqq CE(f, \varrho) \quad (f \in X, \varrho > 0).$$

Lemma 3.2. *Let X be an admissible Banach space (for $j \in \mathbf{P}$) and $\psi \in D^{(j)}$. Then $\beta_\varrho^\psi(x):=\psi(|x|)\lambda_\varrho(x) \in M$ and*

$$(3.6) \qquad \|\beta_\varrho^\psi\|_M \leqq C\psi(\varrho) \quad (\varrho > 0).$$

Proof. Obviously, $v(t):=t\lambda(t) \in BV_{j+1}$ so that Theorem 2.1 yields $v_{1/\psi(2\varrho)}^\psi \in M$, uniformly for $\varrho>0$. In view of the identity

$$\frac{\psi(|x|)}{\psi(2\varrho)} \lambda_\varrho(x) = v_{1/\psi(2\varrho)}^\psi(x)\lambda_\varrho(x),$$

(3.6) holds true since (cf. (2.8)) $\|\beta_\varrho^\psi\|_M \leqq \psi(2\varrho)\|v_{1/\psi(2\varrho)}^\psi\|_M \|\lambda_\varrho\|_M \leqq C\psi(\varrho)$.

Setting $B_\varrho^\psi:=T^{\beta_\varrho^\psi}$ one may now define via $B^\psi g:=\lim_{\varrho\to\infty} B_\varrho^\psi g$ the potential operator B^ψ as a closed, linear operator on the domain

$$(3.7) \qquad X^\psi := \{g \in X;\ \lim_{\varrho\to\infty} \|B_\varrho^\psi g - h\| = 0 \quad \text{for some} \quad h \in X\},$$

called potential space (see [12]). It follows that $\Pi \subset X^\psi$ and

$$(3.8) \qquad B_\varrho^\psi \doteq B^\psi L_\varrho, \quad \|B_\varrho^\psi\|_{[X]} = \|\beta_\varrho^\psi\|_M \leqq C\psi(\varrho).$$

In particular, X^ψ may be equipped with the seminorm $|g|_\psi:=\|B^\psi g\|$ so that the K-functional

$$(3.9) \qquad K_\psi(f, t) := K(X, X^\psi; f, t) := \inf_{g \in X^\psi} \{\|f-g\| + t|g|_\psi\}$$

is well-defined for all $f\in X$, $t\geqq 0$. It defines a seminorm on X for each $t\geqq 0$ and satisfies

$$K_{\psi}(f,t) \leqq \begin{cases} \|f\| & (f\in X) \\ t|f|_{\psi} & (f\in X^{\psi}). \end{cases} \tag{3.10}$$

4. Locally divisible multipliers and Jackson-type inequalities

Throughout the next sections, X is an admissible Banach space for a (fixed) $j\in\mathbf{P}$.

Definition 4.1. Let $\psi\in D^{(j)}$ with inverse function ψ^{-1} and let $\varphi(\varrho)$ be a real-valued, positive function on $\mathcal{J}$. A family of uniformly bounded multipliers $\{\tau_{\varrho}\}_{\varrho\in\mathcal{J}}$ is called locally divisible (at the origin) of order (φ,ψ) if there exists some $\delta>0$ and a uniformly bounded family $\{\theta_{\varrho}\}_{\varrho\in\mathcal{J}}$ of multipliers such that

$$\tau_{\varrho}(x) = \varphi(\varrho)\psi(|x|)\theta_{\varrho}(x) \quad \big(|x| \leqq \psi^{-1}(\delta/\varphi(\varrho))\big). \tag{4.1}$$

If (4.1) holds true for all $x\in\mathbf{R}^N$, $\varrho\in\mathcal{J}$, then the family $\{\tau_{\varrho}\}$ is said to be globally divisible.

Proposition 4.2. *Local divisibility implies the global one of the same order.*

Proof. Let $\{\tau_{\varrho}\}$ satisfy (4.1). Since $1-\lambda(t)=0$ for $0\leqq t\leqq 1$ (see (3.1)), the function $\sigma(t):=(1-\lambda(t))/t$ belongs to BV_{j+1}. Thus $\chi_{\varrho}:=\sigma^{\psi}_{\tilde\varphi(\varrho)}$ and $\nu_{\varrho}:=\lambda^{\psi}_{\tilde\varphi(\varrho)}$ with $\tilde\varphi(\varrho)=(2/\delta)\varphi(\varrho)$ belong to M, uniformly for $\varrho\in\mathcal{J}$ (see Theorem 2.1). Moreover, for all $x\in\mathbf{R}^N$,

$$1-\nu_{\varrho}(x) = \tilde\varphi(\varrho)\psi(|x|)\chi_{\varrho}(x) = (2/\delta)\varphi(\varrho)\psi(|x|)\chi_{\varrho}(x),$$

$$\tau_{\varrho}(x)\nu_{\varrho}(x) = \varphi(\varrho)\psi(|x|)\theta_{\varrho}(x)\nu_{\varrho}(x),$$

$$\tau_{\varrho}(x) = \tau_{\varrho}(x)\nu_{\varrho}(x)+\tau_{\varrho}(x)\big(1-\nu_{\varrho}(x)\big) = \varphi(\varrho)\psi(|x|)[\theta_{\varrho}(x)\nu_{\varrho}(x)+\tau_{\varrho}(x)(2/\delta)\chi_{\varrho}(x)].$$

Hence the assertion follows since the terms in [...] are bounded in M, uniformly for $\varrho\in\mathcal{J}$.

The global and therefore also the local divisibility immediately implies that for any $g\in X^{\psi}$, $\varrho\in\mathcal{J}$ (cf. (3.7, 8))

$$T^{\tau_{\varrho}}g = \lim_{r\to\infty} T^{\tau_{\varrho}}L_r g = \varphi(\varrho)T^{\theta_{\varrho}}\lim_{r\to\infty} B^{\psi}_r g = \varphi(\varrho)T^{\theta_{\varrho}}B^{\psi}g.$$

Thus one obtains (cf. [3], [4])

Theorem 4.3. *Let $\{\tau_{\varrho}\}_{\varrho\in\mathcal{J}}$ be locally divisible of order (φ,ψ). Then there holds true the Jackson-type inequality*

$$\|T^{\tau_{\varrho}}g\| \leqq A_1\varphi(\varrho)|g|_{\psi} \quad (g\in X^{\psi},\ \varrho\in\mathcal{J}), \tag{4.2}$$

and therefore the (global) Jackson-type theorem

$$\|T^{\tau_\varrho}f\| \leqq A_2 K_\psi(f, \varphi(\varrho)) \quad (f\in X, \varrho\in\mathscr{I}). \tag{4.3}$$

Indeed, the estimate (4.3) is an immediate consequence of the definition (3.9) since for any $g\in X^\psi$,

$$\|T^{\tau_\varrho}f\| \leqq \|T^{\tau_\varrho}(f-g)\| + \|T^{\tau_\varrho}g\| \leqq A_0\|f-g\| + A_1\varphi(\varrho)|g|_\psi.$$

A first application yields for the error of the best approximation

Corollary 4.4. *Let* $\psi\in D^{(j)}$. *Then* $E(f, \varrho)\leqq CK_\psi(f, 1/\psi(\varrho))$ $(f\in X, \varrho>0)$.

Proof. Let $\mathscr{I}=(0, \infty)$ and set $v_\varrho:=\lambda^\psi_{\varphi(\varrho)}$, $\varphi(\varrho):=2/\psi(\varrho)$, and $\tau_\varrho:=1-v_\varrho$. Since $\theta(t):=(1-\lambda(t))/t\in BV_{j+1}$ and $\tau_\varrho(x)=(1/\psi(\varrho))\psi(|x|)[2\theta^\psi_{\varphi(\varrho)}(x)]$, the family $\{\tau_\varrho\}$ is globally divisible of order $(1/\psi, \psi)$. Thus Theorem 4.3 implies

$$\|f-T^{v_\varrho}f\| = \|T^{\tau_\varrho}f\| \leqq CK_\psi(f, 1/\psi(\varrho)).$$

Since $v_\varrho(x)=0$ for $|x|\geqq\varrho$, one has $T^{v_\varrho}f\in\Pi_\varrho$ for any $f\in X$ (cf. (3.3)) so that the assertion follows by the definition of $E(f, \varrho)$.

5. Steckin-type inequalities

First observe that in view of (3.4) one has for $p_\varrho\in\Pi_\varrho$, $\varrho>0$,

$$B^\psi_\varrho p_\varrho = B^\psi L_\varrho p_\varrho = B^\psi p_\varrho.$$

Thus Lemma 3.2 implies the following Bernstein-type inequality for polynomials in admissible Banach spaces $(p_\varrho\in\Pi_\varrho, \varrho>0)$

$$|p_\varrho|_\psi = \|B^\psi_\varrho p_\varrho\| \leqq \|B^\psi_\varrho\|_{[X]}\|p_\varrho\| \leqq C\psi(\varrho)\|p_\varrho\|. \tag{5.1}$$

Theorem 5.1. *For* $\psi\in D^{(j)}$ *one has the Steckin-type inequality*

$$K_\psi(f, 1/\varrho) \leqq (C_1/\varrho)\int_0^\varrho E(f, \psi^{-1}(u))\,du \quad (f\in X, \varrho > 0), \tag{5.2}$$

thus for a locally divisible family $\{\tau_\varrho\}_{\varrho\in\mathscr{I}}$ *of order* (φ, ψ)

$$\|T^{\tau_\varrho}f\| \leqq C_2\varphi(\varrho)\int_0^{1/\varphi(\varrho)} E(f, \psi^{-1}(u))\,du. \tag{5.3}$$

Proof. Obviously, (5.3) follows by (5.2) and Theorem 4.3. To show (5.2), set $P_k := L_{\psi^{-1}(2^k)} - L_{\psi^{-1}(2^{k-1})}$, $k \in \mathbf{Z}$. By (3.5) one has

$$\|P_k f\| \leqq \|L_{\psi^{-1}(2^k)} f - f\| + \|L_{\psi^{-1}(2^{k-1})} f - f\| \leqq A_1 E(f, \psi^{-1}(2^{k-1})) \leqq$$
$$\leqq A_1 2^{-k+2} \int_{2^{k-2}}^{2^{k-1}} E(f, \psi^{-1}(u))\, du.$$

Since $P_k f \in \Pi_{2\psi^{-1}(2^k)}$ by (3.3), the Bernstein-type inequality (5.1) yields by (2.8)

$$|P_k f|_\psi \leqq A_2 \psi(2\psi^{-1}(2^k)) \|P_k f\| \leqq A_3 \int_{2^{k-2}}^{2^{k-1}} E(f, \psi^{-1}(u))\, du.$$

In view of (3.8) one has $(k \to -\infty)$

$$\|B^\psi L_{\psi^{-1}(2^k)} f\| = \|B^\psi_{\psi^{-1}(2^k)} f\| \leqq A_4 2^k \|f\| = o(1)$$

so that for $m \in \mathbf{Z}$

$$\sum_{k=-\infty}^{m} B^\psi P_k f = \sum_{k=-\infty}^{m} (B^\psi L_{\psi^{-1}(2^k)} f - B^\psi L_{\psi^{-1}(2^{k-1})} f) = B^\psi L_{\psi^{-1}(2^m)} f.$$

Therefore it follows that

$$|L_{\psi^{-1}(2^m)} f|_\psi \leqq \sum_{k=-\infty}^{m} |P_k f|_\psi \leqq A_3 \sum_{k=-\infty}^{m} \int_{2^{k-2}}^{2^{k-1}} E(f, \psi^{-1}(u))\, du =$$
$$= A_3 \int_0^{2^{m-1}} E(f, \psi^{-1}(u))\, du.$$

Now, let $\varrho > 0$ be arbitrary and $m \in \mathbf{Z}$ be such that $2^m \leqq \varrho < 2^{m+1}$. Then by (3.10)

$$K_\psi(f, 1/\varrho) \leqq K_\psi(f - L_{\psi^{-1}(2^m)} f, 1/\varrho) + K_\psi(L_{\psi^{-1}(2^m)} f, 1/\varrho) \leqq$$
$$\leqq A_4 E(f, \psi^{-1}(2^m)) + (1/\varrho) |L_{\psi^{-1}(2^m)} f|_\psi \leqq$$
$$\leqq A_4 2^{-(m-1)} \int_{2^{m-1}}^{2^m} E(f, \psi^{-1}(u))\, du + (A_3/\varrho) \int_0^{2^{m-1}} E(f, \psi^{-1}(u))\, du \leqq$$
$$\leqq 4(A_4/\varrho) \int_{2^{m-1}}^{\varrho} E(f, \psi^{-1}(u))\, du + (A_3/\varrho) \int_0^{2^{m-1}} E(f, \psi^{-1}(u))\, du.$$

This establishes (5.2) completely.

Let us illustrate Theorem 5.1 in connection with the multiplier criterion of Theorem 2.1.

Corollary 5.2. *Let σ be a complex-valued function on* $[0, \infty)$, *locally divisible (of order $\psi_1(t) = t$) in BV_{j+1}, i.e., there exists an element $\chi \in BV_{j+1}$ satisfying*

$\chi(t)=1$ for $0\leqq t\leqq\delta$ and some $\delta>0$ such that

$$\theta(t) := \sigma(t)t^{-1}\chi(t)\in BV_{j+1}. \tag{5.4}$$

Suppose that $\{\sigma^{\psi}_{\varphi(\varrho)}\}_{\varrho\in\mathcal{J}}$ is of Hardy-type (φ, ψ). Then $\{\sigma^{\psi}_{\varphi(\varrho)}\}$ is globally divisible (in M) of order (φ, ψ). Moreover, there hold true the Jackson- and Steckin-type inequalities ($f\in X$, $\varrho\in\mathcal{J}$)

$$\|T^{\sigma^{\psi}_{\varphi(\varrho)}}f\| \leqq C_1K_{\psi}(f, \varphi(\varrho)) \leqq C_2\varphi(\varrho)\int_0^{1/\varphi(\varrho)} E(f, \psi^{-1}(u))\,du. \tag{5.5}$$

Proof. In view of Theorem 4.3 and 5.1 it is sufficient to prove the local divisibility of $\{\sigma^{\psi}_{\varphi(\varrho)}\}$ (in M) of order (φ, ψ). By (5.4) and Theorem 2.1 the family $\{\theta^{\psi}_{\varphi(\varrho)}\}$ belongs to M, uniformly for $\varrho\in\mathcal{J}$. Moreover,

$$\sigma^{\psi}_{\varphi(\varrho)}(x) = \varphi(\varrho)\psi(|x|)\theta^{\psi}_{\varphi(\varrho)}(x) \quad (\varphi(\varrho)\psi(|x|)\leqq\delta)$$

since $\sigma(t)=t\theta(t)$ for $0\leqq t\leqq\delta$. Hence the assertion follows in view of (4.1).

6. Sharpness of Steckin-type inequalities

Let X^* be the class of bounded, *sub*linear functionals on the Banach space X, endowed with the usual operator norm $\|\cdot\|_{X^*}$. Let ω denote an abstract modulus of continuity, thus a function, continuous on $[0, \infty)$ such that

$$0 = \omega(0) < \omega(t) \leqq \omega(s+t) \leqq \omega(s)+\omega(t) \quad (s, t > 0). \tag{6.1}$$

Additionally, we assume that $\omega(t)\neq O(t)$, i.e.,

$$\lim_{t\to 0+} \omega(t)/t = \infty. \tag{6.2}$$

Moreover, let $\mathcal{J}$ be an unbounded subset of $(0, \infty)$ and φ a positive, monotonically decreasing function on $\mathcal{J}$ satisfying

$$\lim_{\varrho\to\infty} \varphi(\varrho) = 0. \tag{6.3}$$

In these terms one has the following result (see [6], [7]).

Theorem 6.1. *Let φ satisfy* (6.3). *Suppose that for $U_\varrho, V_\varrho\in X^*$ there exist constants C and elements $h_\varrho\in X$ with (r, $\varrho\in\mathcal{J}$)*

$$\|h_\varrho\| \leqq C_1, \tag{6.4}$$

$$\|V_\varrho\|_{X^*} \leqq C_2, \tag{6.5}$$

$$|V_\varrho h_r| \leqq C_3\varphi(\varrho)/\varphi(r), \tag{6.6}$$

$$|U_\varrho h_r| \leqq C_{4,r}\varphi(\varrho), \tag{6.7}$$

$$\liminf_{\varrho\to\infty} |U_\varrho h_\varrho| \geqq C_5 > 0. \tag{6.8}$$

Then for each modulus ω satisfying (6.1, 2) *there exists a counterexample $f_\omega \in X$ such that* $(\varrho \to \infty)$

$$|V_\varrho f_\omega| = O(\omega(\varphi(\varrho))), \quad |U_\varrho f_\omega| \neq o(\omega(\varphi(\varrho))).$$

Suppose that the embedding $M(X) \subset L^\infty(\mathbf{R}^N, E)$, as assumed by the definition, is in fact continuous, i.e.,

$$\|\tau\|_{\infty, E} \leqq C\|\tau\|_M \quad (\tau \in M). \tag{6.9}$$

Corollary 6.2. *Let φ satisfy* (6.3). *Consider a locally divisible family $\{\tau_\varrho\}_{\varrho \in \mathscr{I}} \subset M$ of order (φ, ψ) for which there exist constants K and Borel sets $\{B_\varrho\}_{\varrho \in \mathscr{I}} \subset \Sigma$ with*

$$E(B_\varrho) \neq 0 \quad (\varrho \in \mathscr{I}), \tag{6.10}$$

$$\varphi(\varrho)\psi(|x|) \leqq K_1 \quad (x \in B_\varrho, \varrho \in \mathscr{I}), \tag{6.11}$$

$$|\tau_\varrho(x)| \geqq K_2 > 0 \quad (x \in B_\varrho, \varrho \in \mathscr{I}). \tag{6.12}$$

Then for each modulus (6.1, 2) *there exists a counterexample $f_\omega \in X$ such that* $(\varrho \to \infty)$

$$\varphi(\varrho) \int_0^{\varphi(\varrho)} E(f_\omega, \psi^{-1}(u))\,du = O(\omega(\varphi(\varrho))), \quad \|T^{\tau_\varrho} f_\omega\| \neq o(\omega(\varphi(\varrho))).$$

Proof. Let $\alpha(\varrho) := \psi^{-1}(K_1/\varphi(\varrho))$. For any $B \in \Sigma$ with $E(B) = I$ one has by (2.1) (i), (6.10) that $E(B \cap B_\varrho) = E(B)E(B_\varrho) = E(B_\varrho) \neq 0$, thus $B \cap B_\varrho \neq \emptyset$ by (2.1) (ii). Since $\lambda_{\alpha(\varrho)}(x) = 1$ for $x \in B_\varrho$ (cf. (3.1), (6.11)), it follows by (6.12) that

$$\sup_{x \in B} |\tau_\varrho(x)\lambda_{\alpha(\varrho)}(x)| \geqq \sup_{x \in B \cap B_\varrho} |\tau_\varrho(x)\lambda_{\alpha(\varrho)}(x)| = \sup_{x \in B \cap B_\varrho} |\tau_\varrho(x)| \geqq \inf_{x \in B_\varrho} |\tau_\varrho(x)| \geqq K_2.$$

In view of (2.2) this implies $\|\tau_\varrho \lambda_{\alpha(\varrho)}\|_{\infty, E} \geqq K_2$ and hence $\|\tau_\varrho \lambda_{\alpha(\varrho)}\|_M \geqq K_3 > 0$ by (6.9). Therefore, by the definition of the operator norm (see (2.5)) there exists $f_\varrho \in X$, $\|f_\varrho\| \leqq 1$, such that

$$\|T^{\tau_\varrho} L_{\alpha(\varrho)} f_\varrho\| = \|T^{\tau_\varrho \lambda_{\alpha(\varrho)}} f_\varrho\| \geqq K_4 > 0. \tag{6.13}$$

In order to apply Theorem 6.1 set

$$h_\varrho = L_{\alpha(\varrho)} f_\varrho, \quad V_\varrho f = \varphi(\varrho) \int_0^{1/\varphi(\varrho)} E(f, \psi^{-1}(u))\,du, \quad U_\varrho f = \|T^{\tau_\varrho} f\|.$$

Then $\|h_\varrho\| \leqq K_5$ by (3.2) so that (6.4) is fulfilled. Moreover, (6.5) follows with $C_2 = 1$, and (6.8) coincides with (6.13). It remains to show (6.6) since then (6.7) would also follow by Theorem 5.1 with $C_{4,r}=K_6/\varphi(r)$. But $E(h_r, \psi^{-1}(u))=0$ for $u \geqq \psi(2\alpha(r))$ since $h_r \in \Pi_{2\alpha(r)}$. Thus by (2.8)

$$V_\varrho h_r \leqq \varphi(\varrho) \int_0^{\psi(2\alpha(r))} E(h_r, \psi^{-1}(u))\, du \leqq \psi(2\alpha(r))\varphi(\varrho)\|h_r\| \leqq K_6\varphi(\varrho)/\varphi(r).$$

7. Applications

In this section some applications to the previous abstract results are given by studying concrete examples of spaces H, X, spectral measures E, and processes $\{\tau_\varrho\}$. In Section 7.1 we consider spaces of 2π-periodic functions in connection with one-dimensional trigonometric expansions. It is shown how the present approach covers, in a unified way, those classical results of S. B. Steckin mentioned in Section 1 as well as related material of R. Taberski and M. F. Timan on Abel—Poisson and typical means. In fact, the treatment of this example of a discrete expansion may easily be transferred to other discrete orthogonal systems (Jacobi, Hermite, Laguerre, etc.; for some details see [1a], [4], [8a], [12], [21] and the literature cited there). In Section 7.2 we consider the Abel—Cartwright means in connection with the continuous Fourier spectral measure on the Euclidean N-space, subsuming e.g. results of B. I. Golubov. Finally, Section 7.3 is concerned with a semidiscrete difference scheme for the numerical solution of the heat equation, the results being related to work of G. W. Hedstrom, J. Löfström, J. Peetre, V. Thomée, and others.

7.1. Classical results in spaces of periodic functions. Concerning the spaces $X_{2\pi}$, set $N=1, f_k(x):=e^{ikx}$, and for $B\in\Sigma, f\in L^2_{2\pi}$

$$E(B)f = \sum_{k\in B\cap \mathbf{Z}} f^\wedge(k) f_k.$$

Then E is a spectral measure for the Hilbert space $H=L^2_{2\pi}$. Obviously, $E(B)\neq 0$ iff $B\cap \mathbf{Z}\neq\emptyset$, so that $L^\infty(\mathbf{R}, E)$ may be identified with l^∞, the set of bounded sequences $\{\tau(k)\}_{k\in\mathbf{Z}}\subset\mathbf{C}$. Moreover, $L^2_{2\pi}\cap X_{2\pi}$ is dense in $L^2_{2\pi}$ as well as in $X_{2\pi}$, and the definition (2.4) of a multiplier $\tau=\{\tau(k)\}_{k\in\mathbf{Z}}\subset M(X_{2\pi})$ coincides with the classical one, i.e., for each $f\in X_{2\pi}$ there exists $f^\tau\in X_{2\pi}$ such that $\tau(k)f^\wedge(k)=(f^\tau)^\wedge(k)$ for every $k\in\mathbf{Z}$. Since

$$\|\tau\|_{\infty,E} = \sup_{k\in\mathbf{Z}}|\tau(k)| = \sup_{k\in\mathbf{Z}}\|f_k^\tau\|_{X_{2\pi}} \leqq \|\tau\|_{M(X_{2\pi})},$$

$M(X_{2\pi})$ is continuously embedded in l^∞. By Fejér's theorem, E is (R, j)-bounded on any $X_{2\pi}$ for $j=1$ (at least). Moreover, Π_ϱ coincides with (1.1), and Π is dense in $X_{2\pi}$ so that all the Banach spaces $X_{2\pi}$ are admissible (for details cf. [11], [12]).

Concerning the rth modulus (1.3) of continuity ($r\in\mathbf{N}$ even), Theorem 5.1 delivers Steckin's result (1.4).

Corollary 7.1. *Let* $r\in\mathbf{N}$ *be even. Then there holds true the inequality* ($f\in X_{2\pi}$, $n\in\mathbf{N}$)

$$\omega_r(X_{2\pi};\, f, 1/n) \leqq C_1 n^{-r} \int_0^{n^r} E(X_{2\pi};\, f, u^{1/r})\, du \leqq \tag{7.1}$$

$$\leqq C_2 n^{-r} \sum_{j=0}^{n-1} (j+1)^{r-1} E(X_{2\pi};\, f, j).$$

Proof. Let $\mathscr{I}=\mathbf{N}$ and $\tau_n^r(k):=(1-e^{ik/n})^r$. Since $\sigma^r(t)=(1-e^{it})^r/t^r$ is infinitely differentiable on $\mathbf{R}$, the multipliers $\theta_n(k):=\sigma^r(k/n)\lambda_n(k)$ belong to $M(X_{2\pi})$, uniformly for $n\in\mathbf{N}$. Now $\tau_n^r(k)=(k/n)^r\theta_n(k)$ for $|k|\leqq n$, so that $\{\tau_n^r\}$ is locally divisible of order (φ_r, ψ_r) with $\psi_r(u)=u^r$, $\varphi_r(n)=n^{-r}$. Then the first inequality of (7.1) is a consequence of (5.3), whereas the second one follows by substituting $u^{1/r}=t$ and using the monotonicity of $E(f, t)$.

Note that Theorem 5.1 is not applicable for odd $r\in\mathbf{N}$ since the corresponding potential multiplier $(ik)^r$ is not radial.

To reproduce Steckin's second result on Fejér sums, let us introduce a more general class of operators, the typical means

$$Z_n^r f := \sum_{k=-n}^{n} \left(1-\left(\frac{|k|}{n+1}\right)^r\right) f^\frown(k) f_k.$$

Corollary 7.2. *For* $r\in\mathbf{N}$, $n\in\mathbf{P}$, *and* $f\in X_{2\pi}$

$$\|Z_n^r f - f\|_{X_{2\pi}} \leqq C_1 (n+1)^{-r} \int_0^{(n+1)^r} E(X_{2\pi};\, f, u^{1/r})\, du \leqq \tag{7.2}$$

$$\leqq C_2 (n+1)^{-r} \sum_{j=0}^{n} (j+1)^{r-1} E(X_{2\pi};\, f, j).$$

On the other hand, for each modulus (6.1, 2) *there exists an element* $f_\omega\in X_{2\pi}$ *such that* ($t\to 0+$, $n\to\infty$)

$$t \int_0^{1/t} E(X_{2\pi};\, f_\omega, u^{1/r})\, du = O(\omega(t)), \tag{7.3}$$

$$\|Z_n^r f_\omega - f_\omega\|_{X_{2\pi}} \neq o(\omega((n+1)^{-r})). \tag{7.4}$$

Proof. For an application of Corollary 5.2, set $\mathscr{I}=\mathbf{P}$, and $\sigma(t)=1-(1-t)_+$, $\psi(t)=t^r$, $\varphi(n)=(n+1)^{-r}$. Then the multiplier $\sigma_{\varphi(n)}^{\psi}$ of Hardy-type (φ, ψ) corresponds to the remainder $I-Z_n^r$. Since $\theta(t):=\sigma(t)\lambda(t)/t\in BV_2$, condition (5.4) is fulfilled so that (7.2) follows by (5.5). Concerning the sharpness of (7.2), apply Corollary 6.2 with $B_n=\{n+1\}$. Since (6.10—12) follow with $K_1=K_2=1$, one obtains (7.3, 4) at once.

Obviously, (7.2) for $r=1$ regains Steckin's result (1.5) on the Fejér means whereas for $r>1$ inequality (7.2) was established in [20]. Concerning the sharpness, it was shown in [6], [7] that even

$$\limsup_{n\to\infty} \|F_n f_\omega - f_\omega\|_{X_{2\pi}}/E(X_{2\pi};\ f_\omega, n) = \infty$$

for some element f_ω satisfying (7.3, 4).

Concerning the Abel—Poisson means, given by $(r\in(0,1)=\mathcal{I}, f\in X_{2\pi})$

$$P_r f := \sum_{k=-\infty}^{\infty} r^{|k|} f^\wedge(k) f_k;$$

consider the multiplier $p_r(k):=r^{|k|}=e^{-|k|\,|\log r|}$ of Hardy-type (φ, ψ) with $\psi(u)=u$, $\varphi(r)=|\log r|$. Since $(1-e^{-t})$, $(1-e^{-t})\lambda(t)/t\in BV_2$, Corollary 5.2 delivers (cf. [18], [20])

Corollary 7.3. *For the Abel—Poisson means P_r one has the Steckin-type inequality* $(f\in X_{2\pi}, 0<r<1)$

$$\|P_r f - f\|_{X_{2\pi}} \le C|\log r| \int_0^{1/|\log r|} E(X_{2\pi};\ f, u)\,du \le C\frac{1-r}{r} \sum_{0\le j\le 1/(1-r)} E(X_{2\pi};\ f, j).$$

7.2. Abel—Cartwright means in $L^p(\mathbf{R}^N)$. Let $L^p=L^p(\mathbf{R}^N)$, $1\le p\le\infty$, $N\in\mathbf{N}$, be the space of Lebesgue measurable functions on $\mathbf{R}^N$ for which the norm

$$\|f\|_p := \begin{cases} \left\{(2\pi)^{-N/2} \int_{\mathbf{R}^N} |f(u)|^p\,du\right\}^{1/p} & (1\le p<\infty) \\ \operatorname{ess\,sup}_{u\in\mathbf{R}^N} |f(u)| & (p=\infty), \end{cases}$$

respectively, is finite. For $f\in L^2$ let $\mathcal{F}f:=f^\wedge$ be the Fourier—Plancherel transform of f:

$$\lim_{\varrho\to\infty} \|(2\pi)^{-N/2} \int_{|u|\le\varrho} f(u)e^{-ivu}\,du - f^\wedge(v)\|_2 = 0,$$

and $\mathcal{F}^{-1}$ the inverse operator. For $B\in\Sigma$ let $\mathcal{P}_B$ be the multiplication projection:

$$\mathcal{P}_B f := \varkappa_B f,\quad \varkappa_B(u) := \begin{cases} 1, & u\in B, \\ 0, & u\notin B. \end{cases}$$

Then $E(B):=\mathcal{F}^{-1}\mathcal{P}_B\mathcal{F}$ is a spectral measure for the Hilbert space $H=L^2$, and $L^\infty(\mathbf{R}^N, E)=L^\infty$ (cf. [8, p. 1989]). Furthermore, $X=L^p$ satisfies (2.3) for $1\le p<\infty$, and (2.4) coincides with the classical definition of Fourier multipliers, i.e., $\tau\in M_p:=$

$:=M(L^p)$ iff $T^\tau f:=\mathcal{F}^{-1}(\tau f^\wedge)\in L^p$, $\|T^\tau f\|_p\leqq A\|f\|_p$ for any $f\in L^2\cap L^p$ (cf. [17, p. 94]). Moreover, $M_p\subset L^\infty$ continuously. Note that E is (R,j)-bounded for L^p if, e.g., $j>(N-1)|1/p-1/2|$ (cf. [17, p. 114]), and that the polynomials are dense in L^p, where $\Pi_\varrho=\Pi_{\varrho,p}$ is the set of entire functions on $\mathbf{C}^N$ of (radial) exponential type ϱ the restriction to $\mathbf{R}^N$ of which belongs to L^p. Thus, the spaces L^p are admissible for $1\leqq p<\infty$.

Let $\psi\in D^{(j)}$ for some $j>(N-1)|1/p-1/2|$ and $\varphi(t)>0$ for $t>0$. Consider the (generalized) Abel—Cartwright means $W^\psi_{\varphi(t)}$, corresponding to the multiplier $w^\psi_{\varphi(t)}$, $w(u)=e^{-u}$, of Hardy-type (φ,ψ). Since $w\in BV_{j+1}$ for every $j\in\mathbf{P}$, the approximation process $W^\psi_{\varphi(t)}$ is well-defined in $[L^p]$, uniformly bounded for $t>0$. In particular, $\psi(u)=\varphi(u)=u^\alpha$, $\alpha>0$, yields the standard Abel—Cartwright means $W_\alpha(t)$ which subsume for $\alpha=1$ the Abel—Poisson and for $\alpha=2$ the Gauss—Weierstrass means.

Corollary 7.4. *Let $1\leqq p<\infty$ and $j>(N-1)|1/p-1/2|$. Suppose that $\psi\in D^{(j)}$, and let $\varphi(t)$ be a positive function, tending monotonically to zero for $t\to 0+$* (cf. (6.3)). *Then* $(f\in L^p(\mathbf{R}^N),\ t>0)$

$$\text{(7.5)}\qquad \|W^\psi_{\varphi(t)}f-f\|_p\leqq C_1K_\psi\big(f,\varphi(t)\big)\leqq C_2\varphi(t)\int_0^{1/\varphi(t)}E\big(L^p;\ f,\psi^{-1}(u)\big)\,du.$$

On the other hand, for each modulus (6.1, 2) *there exists a counterexample $f_\omega\in L^p$ such that* $(t\to 0+)$

$$\text{(7.6)}\qquad \varphi(t)\int_0^{1/\varphi(t)}E\big(L^p;\ f_\omega,\psi^{-1}(u)\big)\,du=O\big(\omega(\varphi(t))\big),$$

$$\text{(7.7)}\qquad \|W^\psi_{\varphi(t)}f_\omega-f_\omega\|_p\neq o\big(\omega(\varphi(t))\big).$$

Proof. Obviously, (7.5) follows by Corollary 5.2 since $(1-e^{-u})/u\in BV_{j+1}$ for every $j\in\mathbf{P}$. Concerning the sharpness of (7.5), set $\varrho=1/t$, $B_\varrho=\{x\in\mathbf{R}^N;\ 1\leqq\varphi(1/\varrho)\psi(|x|)\leqq 2\}$, and $\tau_\varrho=1-w^\psi_{\varphi(1/\varrho)}$. Then (6.10—12) follow with $K_1=2$, $K_2=1-e^{-1}$ so that Corollary 6.2 delivers (7.6, 7).

Let us consider the rate of convergence of the standard Abel—Cartwright means $W_\alpha(t)f$ for elements $f\in L^p$ belonging to the (radial (cf. (1.3))) Lipschitz classes $(k\in\mathbf{N},\ 0<\beta\leqq 2k)$

$$\mathrm{Lip}_{2k}\big(L^p(\mathbf{R}^N);\ \beta\big):=\big\{f\in L^p(\mathbf{R}^N);\ \omega_{2k}\big(L^p(\mathbf{R}^N);\ f,t\big)=O(t^\beta);\ t\to 0+\big\}.$$

Since one has $(\psi_{2k}(u)=u^{2k}$; cf. [24])

$$\text{(7.8)}\qquad K_{\psi_{2k}}(f,t^{2k}):=K\big(L^p,(L^p)^{\psi_{2k}};\ f,t^{2k}\big)\leqq C_k\omega_{2k}\big(L^p(\mathbf{R}^N);\ f,t\big),$$

Corollary 7.4 delivers (cf. [9])

Corollary 7.5. *Let $k\in\mathbf{N}$, $0<\alpha,\beta\leqq 2k$, and $f\in\mathrm{Lip}_{2k}(L^p;\beta)$.*

(i) *If* $0<\alpha<2k$, *then* $(t\to 0+)$

$$\|W_\alpha(t)f-f\|_p = \begin{cases} O(t^\beta), & 0<\beta<\alpha, \\ O(t^\alpha|\log t|), & \beta=\alpha, \\ O(t^\alpha), & \beta>\alpha. \end{cases} \tag{7.9}$$

(ii) *For* $\alpha=2k$ *one has*

$$\|W_\alpha(t)f-f\|_p = O(t^\beta). \tag{7.10}$$

Proof. Obviously, (7.10) follows by (7.5, 8). Concerning (7.9), Corollary 7.4 implies (cf. [19a] for $a=1$)

$$\|W_\alpha(t)f-f\|_p = O\Big(t^\alpha \int_0^{t^{-\alpha}} E(L^p;\, f, u^{1/\alpha})\,du\Big). \tag{7.11}$$

By Corollary 4.4 and (7.8) the assumption yields $(u\to\infty)$

$$E(f, u^{1/\alpha}) = O\big(K_{\psi_{2k}}(f, u^{-2k/\alpha})\big) = O\big(\omega_{2k}(L^p;\, f, u^{-1/\alpha})\big) = O(u^{-\beta/\alpha}),$$

and the assertion follows by (7.11).

7.3. A semidiscrete difference scheme for the heat equation. In the frame of Section 7.2, let $N=1$ and $1\leq p<\infty$. In order to approximate the exact solution of the heat equation $(x\in\mathbf{R},\, t>0)$

$$d/dt\, u(x,t) = d^2/dx^2\, u(x,t), \quad u(x,0) = f(x)\in L^p,$$

given by the Gauss—Weierstrass means

$$W_2(t^{1/2})f(x) := (4\pi t)^{-1/2} \int_{-\infty}^{\infty} f(x-u)e^{-u^2/4t}\,du,$$

consider the initial value problem for $h>0$

$$d/dt\, u_h(x,t) = h^{-2}[u_h(x+h,t)-2u_h(x,t)+u_h(x-h,t)],\quad u_h(x,0) = f(x).$$

This leads to the semidiscrete difference scheme (cf. [2])

$$u_h(\cdot\,,t) := D_h(t)f := T^{d_{h,t}}f, \quad d_{h,t}(x) := e^{-2(t/h^2)(1-\cos(xh))}.$$

Thus the multiplier $\tau_{h,t}$ of the remainder $D_h(t)-W_2(t^{1/2})$ has the representation

$$\tau_{h,t}(x) := g_{t/h^2}(xh), \quad g_r(u) := e^{-2r(1-\cos u)} - e^{-ru^2}. \tag{7.12}$$

Lemma 7.6. *The family* $\{g_r\}_{r>0}$ *is globally divisible of order* (φ_1, ψ_2) *with* $\varphi_1(r)=r$, $\psi_2(u)=u^2$ *and satisfies the (local) condition*

$$g_r(u) = ru^4e^{-aru^2}\theta_r(u) \quad (|u|\leq\delta) \tag{7.13}$$

for $a=2/\pi^2$, $\delta=\pi/2$, *where the family* $\{\theta_r\}\subset M_p$ *is uniformly bounded for* $r>0$. *Moreover, there exists a constant* $c>0$ *such that*

$$g_r(u)\geq c \quad \big(r(u-2\pi)^2\leq 1,\, (2\pi)^2r\geq 9\big). \tag{7.14}$$

Proof. Since $e^{-ru^2}\in M_p$, uniformly in $r>0$ (cf. Theorem 2.1), and

$$\|e^{-2r(1-\cos u)}\|_{M_p} \leqq e^{-2r}\sum_{k=0}^{\infty}\frac{1}{k!}\|2r\cos u\|_{M_p}^k \leqq 1, \tag{7.15}$$

the family $\{g_r\}$ is uniformly bounded in M_p for $r>0$. To show (4.1), the Fejér-kernel $\sigma(u):=2u^{-2}(1-\cos u)$ belongs to M_p as well as $\chi(u):=(1-\sigma(u))/u^2$ (cf. Theorem 2.1). Consider the identity

$$g_r(u) = ru^2(1-\sigma(u))\int_0^1 e^{-(1-s)ru^2}e^{-2sr(1-\cos u)}\,ds. \tag{7.16}$$

Since the integral is uniformly bounded in M_p (cf. (7.15)), one has global divisibility of $\{g_r\}$ of order (φ_1, ψ_2). Concerning (7.13), set $e_r(x):=\exp\left[r(ax^2-2(1-\cos x))\right]$. Since $1-\cos x\geqq ax^2$ for $|x|\leqq 2\delta$, one has for $|x|\leqq 2\delta$:

$$0\leqq e_r(x)\leqq 1,\quad |e_r'(x)|\leqq C_1r|x|e^{-arx^2},\quad |e_r''(x)|\leqq C_2r(1+rx^2)e^{-arx^2}.$$

In view of (3.1) it follows that

$$\|e_r\lambda_\delta\|_{BV_2}\leqq \int_0^{2\delta} x[|e_r(x)\lambda_\delta''(x)|+2|e_r'(x)\lambda_\delta'(x)|+|e_r''(x)\lambda_\delta(x)|]\,dx\leqq$$

$$\leqq \int_0^{2\delta} x|\lambda_\delta''(x)|\,dx+2C_1\sup_{x\geqq 0}|x\lambda_\delta'(x)|\int_0^{2\delta} rxe^{-arx^2}\,dx+$$

$$+C_2\int_0^{2\delta} rx(1+rx^2)e^{-arx^2}\,dx\leqq C_3<\infty.$$

Thus $e_r\lambda_\delta\in M_p$, uniformly for $r>0$ (cf. Theorem 2.1). Therefore one obtains by (7.16) that for $|u|\leqq\delta$

$$g_r(u) = ru^4\chi(u)e^{-aru^2}\int_0^1 e^{-r(1-s)(1-a)u^2}e_{sr}(u)\lambda_\delta(u)\,ds.$$

Hence (7.13) follows since

$$\Big\|\int_0^1 e^{-r(1-s)(1-a)u^2}e_{sr}(u)\lambda_\delta(u)\,ds\Big\|_M\leqq C\|e^{-u^2}\|_{BV_2}\sup_{r>0}\|e_r\lambda_\delta\|_{BV_2}.$$

Finally, let $(2\pi)^2r\geqq 9$ and $(u-2\pi)^2\leqq 1/r$. Then $u^2\geqq 4/r$ and

$$g_r(u) = e^{-2r(1-\cos(u-2\pi))}-e^{-ru^2}\geqq e^{-r(u-2\pi)^2}-e^{-4}\geqq e^{-1}-e^{-4}=:c>0.$$

Corollary 7.7. *For* $f\in L^p(\mathbf{R})$, $1\leqq p<\infty$, *and* $h, t>0$

$$\|D_h(t)f-W_2(t^{1/2})f\|_p\leqq C_1K(L^p,(L^p)^{\psi_2};f,h^2)\leqq C_2h^2\int_0^{h^{-2}}E(L^p;f,u^{1/2})\,du. \tag{7.17}$$

On the other hand, for each (fixed) $t>0$ and each modulus (6.1, 2) *there exists a counterexample $f_\omega \in L^p$ such that* $(h\to 0+)$

$$h^2 \int_0^{h^{-2}} E(L^p;\, f_\omega, u^{1/2})\, du = O(\omega(h^2)), \quad \|D_h(t) f_\omega - W_2(t^{1/2}) f_\omega\|_p \neq o(\omega(h^2)).$$

Proof. Set $\mathscr{I} = \{(h, t);\, h, t>0\}$ and $\varphi(h, t) = h^2$. Since M_p is dilation-invariant, i.e., $\|\chi(xh)\|_M = \|\chi\|_M$, the uniform boundedness of $\{g_r\}$ implies that of $\{\tau_{h,t}\}$ (cf. (7.12)). In view of (7.13) one has

$$\tau_{h,t}(x) = g_{t/h^2}(xh) = h^2 x^2 t x^2 e^{-atx^2} \theta_{t/h^2}(xh) \quad (|xh| \leqq \delta),$$
$$\|t x^2 e^{-atx^2} \theta_{t/h^2}(xh)\|_{M_p} \leqq \|t x^2 e^{-atx^2}\|_{M_p} \|\theta_{t/h^2}\|_{M_p} \leqq K$$

(cf. Theorem 2.1). This implies the local divisibility of order (φ, ψ_2), and thus (7.17) by Theorems 4.3, 5.1.

In order to apply Corollary 6.2, set $\varrho = 1/h$, $\varphi(\varrho) = 1/\varrho^2$, and $B_\varrho = \{x \in \mathbf{R};\, t(x - 2\pi\varrho)^2 \leqq 1\}$. Then (6.10, 11) follow at once, and (6.12) by (7.14) for $\varrho \geqq 3/2\pi t^{1/2}$.

In view of (7.8, 17) one has

$$\|D_h(t) f - W_2(t^{1/2}) f\|_p \leqq C\omega_2(L^p(\mathbf{R});\, f, h);$$

uniformly for $t>0$. This estimate can be improved to the following one which reflects the behaviour for e.g. $t\to 0+$ more precisely.

Corollary 7.8. *For $f \in L^p(\mathbf{R})$ and $h, t>0$*

$$\|D_h(t) f - W_2(t^{1/2}) f\|_p \leqq C \begin{cases} (h^2/t)\,\omega_4(L^p;\, f, t^{1/2}) + E(L^p;\, f, \delta/2h), \\ \omega_2(L^p;\, f, t^{1/2}). \end{cases} \tag{7.18}$$

Proof. Apply Theorem 4.3 to $\sigma_t(x) = a^2 t^2 x^4 e^{-atx^2}$ which belongs to M_p, uniformly for $t>0$, since $u^4 \exp(-u^2) \in BV_2$. Obviously, it is globally divisible of order (φ_2, ψ_4) so that Theorem 4.3 and (7.8) imply

$$\|T^{\sigma_t} f\|_p \leqq A_1 K_{\psi_4}(f, t^2) \leqq A_2\, \omega_4(L^p;\, f, t^{1/2}). \tag{7.19}$$

In view of (7.13) one has for all $x \in \mathbf{R}$

$$\tau_{h,t}(x) = \tau_{h,t}(x) \lambda_{\delta/2h}(x) + \tau_{h,t}(x)(1 - \lambda_{\delta/2h}(x)) =$$
$$= (h^2/t)\, \sigma_t(x) [a^{-2} \theta_{t/h^2}(xh) \lambda_{\delta/2h}(x)] + \tau_{h,t}(x)(1 - \lambda_{\delta/2h}(x)).$$

Hence the first inequality follows by (3.5) and (7.19).

Since $\{g_r\}$ is globally divisible of order (φ_1, ψ_2) (cf. Lemma 7.6), there exists $\{v_r\}_{r>0} \subset M_p$, uniformly bounded for $r>0$, such that $g_r(u) = r u^2 v_r(u)$. Hence $\tau_{h,t}(x) = g_{t/h^2}(xh) = t x^2 v_{t/h^2}(xh)$ so that the second part of (7.18) follows by Theorem 4.3 and (7.8).

In Chapter IV of [2] (see also the literature cited there), the fundamental telescoping technique was also used for a parallel treatment of the present example in order to obtain error bounds on Besov spaces. The approach of this paper, however, uses the same technique only in the abstract setting in order to derive the estimates of Theorem 5.1. Consequently, for the concrete example one only needs to verify the basic divisibility assumptions. This procedure in fact delivers a comparison of the processes on the whole space.

References

[1] G. Bragard and R. J. Nessel, A local comparison theorem with applications to approximation in certain group and multiplier algebras, *Anal. Math.*, **2** (1976), 87—98.

[1a] H. Brass, Approximation mittels Linearkombinationen von Projektionsoperatoren, *Abh. Braunschweig. Wiss. Ges.*, **18** (1966), 50—69.

[2] P. Brenner, V. Thomée and L. Wahlbin, *Besov Spaces and Applications to Difference Methods for Initial Value Problems*, Lecture Notes in Mathematics 434, Springer (Berlin, 1975).

[3] P. L. Butzer and J. Junggeburth, On Jackson-type inequalities in approximation theory, in: *General Inequalities I* (Proc. Conf. Oberwolfach 1976, E. F. Beckenbach, Ed.), ISNM 41, Birkhäuser (Basel, 1978).

[4] P. L. Butzer, R. J. Nessel and W. Trebels, Multipliers with respect to spectral measures in Banach spaces and approximation, I: Radial multipliers in connection with Riesz-bounded spectral measures, *J. Approximation Theory*, **8** (1973), 335—356.

[5] R. A. DeVore, S. D. Riemenschneider and R. C. Sharpley, Weak interpolation in Banach spaces, *J. Functional Analysis*, **33** (1979), 58—94.

[6] W. Dickmeis, R. J. Nessel and E. van Wickeren, On the sharpness of estimates in terms of averages, *Math. Nachr.*, **117** (1984), in print.

[7] W. Dickmeis, R. J. Nessel and E. van Wickeren, *A general approach to quantitative negative results in approximation theory*, Publ. House Bulgarian Acad. Sci. (Sofia, 1984) (in print).

[8] N. Dunford and J. T. Schwartz, *Linear Operators*. Vol. I: *General Theory*, Vol. II: *Spectral Theory*, Vol. III: *Spectral Operators*, Interscience (New York, 1958, 1963, 1971).

[8a] G. Freud, Über die (*C*, 1)-Summen der Entwicklungen nach orthogonalen Polynomen, *Acta Math. Acad. Sci. Hungar.*, **14** (1963), 197—208.

[9] B. I. Golubov, On convergence of singular integrals of Gauss—Weierstrass type for functions of several variables, *Dokl. Akad. Nauk SSSR*, **248** (1979), 1044—1048. (Russian); *Soviet Math. Dokl.*, **20** (1979), 1103—1107.

[10] J. Löfström, Besov spaces in theory of approximation, *Ann. Mat. Pura Appl.*, **85** (1970), 93—184.

[11] R. J. Nessel and G. Wilmes, Nikolskiĭ-type inequalities in connection with regular spectral measures, *Acta Math. Acad. Sci. Hungar.*, **33** (1979), 169—182.

[12] R. J. Nessel und G. Wilmes, *Über Ungleichungen vom Bernstein—Nikolskiĭ—Riesz-Typ in Banach Räumen*, Forschungsberichte des Landes Nordrhein-Westfalen 2841, Westdeutscher Verlag (Opladen, 1979).

[13] J. Peetre, *New Thoughts on Besov Spaces*, Duke University (Durham, 1976).

[14] H. S. Shapiro, *Topics in Approximation Theory*, Lecture Notes in Mathematics 187, Springer (Berlin, 1971).

[15] S. B. Steckin, On the order of best approximation of continuous functions, *Izv. Akad. Nauk SSSR Ser. Mat.*, **15** (1951), 219—242. (Russian).

[16] S. B. Steckin, The approximation of periodic functions by Fejér sums, *Trudy Mat. Inst. Steklov*, **62** (1961), 48—60. (Russian); *Amer. Math. Soc. Transl.*, (2) **28** (1963), 269—282.

[17] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press (Princeton, 1970).

[18] R. Taberski, A theorem of Steckin and Leindler type connected with Abel summability of Fourier series, *Demonstratio Math.*, **8** (1975), 215—225.

[19] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press (New York, 1963).

[19a] M. F. Timan, Deviation of harmonic functions from their boundary values and best approximation, *Dokl. Akad. Nauk. SSSR*, **145** (1962), 1008—1009. (Russian); *Soviet Math. Dokl.*, **3** (1962), 1157—1159.

[20] M. F. Timan, The best approximation of a function and linear methods for the summation of Fourier series, *Izv. Akad. Nauk SSSR Ser. Mat.*, **29** (1965), 587—604. (Russian); *Amer. Math. Soc. Transl.*, (2) **77** (1968), 127—147.

[21] W. Trebels, *Multipliers for (C, α)-Bounded Fourier Expansions in Banach Spaces and Approximation*, Lecture Notes in Mathematics 329, Springer (Berlin, 1973).

[22] W. Trebels, Some Fourier multiplier criteria and the spherical Bochner—Riesz kernel, *Rev. Roumaine Math. Pures Appl.*, **20** (1975), 1173—1185.

[23] H. Triebel, *Fourier Analysis and Function Spaces*, Teubner (Leipzig, 1977).

[24] G. Wilmes, On Riesz-type inequalities and K-functionals related to Riesz potentials in $\mathbf{R}^N$, *Numer. Funct. Anal. Optim.*, **1** (1979), 57—77.

LEHRSTUHL A FÜR MATHEMATIK
RWTH AACHEN
TEMPLERGRABEN 55
5100 AACHEN, GERMANY

Acta Sci. Math., 47 (1984), 189—200

Unbedingte Konvergenz der Orthogonalreihen

KÁROLY TANDORI

1. In der Arbeit [1] haben wir den folgenden Satz bewiesen.

Satz A. *Es sei* $\{a_n\}_1^\infty$ *eine monoton abnehmende Folge von positiven Zahlen mit*

$$\sum_{\nu=0}^{\infty} \sqrt{\sum_{n=2^{2^\nu}+1}^{2^{2^{\nu+1}}} a_n^2 \log^2 n} = \infty. \tag{1}$$

Dann gibt es ein orthonormiertes System $\{\varphi_n(x)\}_1^\infty$ *der Treppenfunktionen im Interval* (0, 1) *derart, dass die Reihe*

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \tag{2}$$

eine Anordnung

$$\sum_{k=1}^{\infty} a_{n_k} \varphi_{n_k}(x) \tag{3}$$

ihrer Glieder besitzt, die fast überall in (0, 1) *divergiert.*

In dieser Note werden wir einen ziemlich einfacheren Beweis auf diesen Satz geben.

Es sei $N_\nu = 2+2^2+\ldots+2^{2^\nu}$ $(\nu=0, 1, \ldots)$. Es ist klar, dass für eine monoton abnehmende Folge $\{a_n\}_1^\infty$ von positiven Zahlen die Bedingung (1) mit der Bedingung

$$\sum_{\nu=0}^{\infty} \sqrt{\sum_{n=N_\nu+1}^{N_{\nu+1}} a_n^2 \log^2 n} = \infty \tag{4}$$

äquivalent ist. Weiterhin aus (4) folgt:

$$\sum_{\nu=0}^{\infty} \sqrt{\sum_{n=N_\nu+1}^{N_{\nu+1}} \bar{a}_n^2 \log^2 n} = \infty,$$

Eingegangen am 8. December 1982.

wobei

$$\bar{a}_n = a_{N_\nu + 2^{2^\nu}+1} \quad (N_\nu < n \leqq N_\nu + 2\cdot 2^{2^\nu};\ \nu = 0, 1, \ldots),$$

$$\bar{a}_n = a_{N_\nu + 2^{2^\nu+s}+1}$$

$$(N_\nu + 2^s\cdot 2^{2^\nu} < n \leqq N_\nu + 2^{s+1}\cdot 2^{2^\nu};\ s = 1, \ldots, 2^\nu - 1,\ \nu = 1, 2, \ldots)$$

ist. Für diese Folge gilt $\bar{a}_n \leqq a_n$ $(n=3, 4, \ldots)$.

Nach Obigen und nach einem bekannten Satz (s. [2], Satz III) ist es genug den folgenden Satz zu beweisen.

Satz B. *Es sei* $\{a_n\}_1^\infty$ *eine monoton abnehmende Folge von positiven Zahlen mit* (4), *für die*

$$a_n = a_{N_\nu + 2^{2^\nu}+1} \quad (N_\nu < n \leqq N_\nu + 2\cdot 2^{2^\nu};\ \nu = 0, 1, \ldots);$$

$$a_n = a_{N_\nu + 2^{2^\nu+s}+1} \tag{5}$$

$$(N_\nu + 2^s\cdot 2^{2^\nu} < n \leqq N_\nu + 2^{s+1}\cdot 2^{2^\nu};\ s = 1, \ldots, 2^\nu - 1;\ \nu = 1, 2, \ldots)$$

ist. Dann gibt es ein orthonormiertes System $\{\varphi_n(x)\}_1^\infty$ *von Treppenfunktionen in* (0, 1) *derart, dass die Reihe* (2) *eine Anordnung* (3) *ihrer Glieder besitzt, die fast überall in* (0, 1) *divergiert.*

2. Zum Beweis des Satzes B benötigen wir gewisse Hilfssätze.

Für eine Folge $b = \{b_n\}_1^\infty$ setzen wir

$$\|b\| = \sup_{\varphi} \sqrt{\int_0^1 \sup_{1\leqq i\leqq j} \left(\sum_{n=i}^{j} b_n \varphi_n(x)\right)^2 dx},$$

wobei das Supremum für jedes orthonormierte System $\varphi = \{\varphi_n(x)\}_1^\infty$ in (0, 1) gebildet wird. $C_1, C_2, \ldots$ bezeichnen positive Konstante.

Hilfssatz I. *Für jede Folge* $b = \{b_n\}_1^\infty$ *mit* $|b_n| \geqq |b_{n+1}|$ $(n = 1, 2, \ldots)$ *gilt*

$$\|b\| \geqq C_1 \left(b_1^2 + \sum_{n=2}^{\infty} b_n^2 \log^2 n\right)^{1/2} \quad (C_1 \leqq 1).$$

Hilfssatz I ist bekannt. (S. [2], Satz VII.)

Hilfssatz II. *Für jede Folge* $b = \{b_1, \ldots, b_N, 0, \ldots\}$ *gibt es ein orthonormiertes System* $\{\Psi_n(x)\}_1^N$ *der Treppenfunktionen in* (0, 1) *und eine einfache Menge* E $(\subseteqq (0, 1))$

derart, dass

$$\max_{1\leqq m\leqq N}\left(b_1\Psi_1(x)+\dots+b_m\Psi_m(x)\right)\geqq C_2\|b\|\quad (x\in E)\quad (C_2\leqq 1), \tag{6}$$

$$\operatorname{mes} E\geqq C_3\quad (C_3\leqq 1) \tag{7}$$

erfüllt sind.

(Eine Menge wird einfach genannt, wenn sie die Vereinigung endlichvieler Intervalle ist.)

Hilfssatz II ist eine einfache Folgerung von [2], Hilfssatz VIII.

Hilfssatz III. *Es seien b eine Zahlenfolge und N eine positive ganze Zahl derart, dass*

$$\sum_{n=1}^{N} b_n^2\leqq \frac{C_2^2 C_3}{8}\|b\|^2. \tag{8}$$

Dann gibt es ein orthonormiertes System $\{\Psi_n(x)\}_1^N$ *der Treppenfunktionen in* $(0, 1)$ *und paarweise disjunkte Intervalle* $I_1, \dots, I_N$ $\left(\subseteqq(0, C_3/8)\right)$ *mit* $\operatorname{mes} I_i=C_3/8N\ (i=1, \dots, N)$ *derart, dass mit gewissen Indizes* m_i $(1\leqq m_i\leqq N)$

$$\left.\begin{aligned} b_1\Psi_1(x)+\dots+b_{m_i}\Psi_{m_i}(x)&\geqq\frac{C_2}{2}\|b\|,\\ b_{m_i+1}\Psi_{m_i+1}(x)+\dots+b_N\Psi_N(x)&\leqq-\frac{C_2}{2}\|b\|\end{aligned}\right\}\quad (x\in I_i)$$

$(i=1, \dots, N)$ *erfüllt sind.*

Beweis des Hilfssatzes III. Durch Anwendung des Hilfssatzes II gibt es ein orthonormiertes System $\{\psi_n(x)\}_1^N$ der Treppenfunktionen in $(0, 1)$ und eine einfache Menge $E(\subseteqq(0, 1))$ mit (6) und (7). Es sei

$$G=\left\{x\in(0, 1)\colon \left|\sum_{n=1}^{N} b_n\Psi_n(x)\right|\geqq\frac{C_2}{2}\|b\|\right\}.$$

G ist einfach und nach der Tschebyschevschen Ungleichung folgt

$$\operatorname{mes} G\leqq 4\,\frac{\sum_{n=1}^{N} b_n^2}{C_2^2\|b\|^2}\leqq\frac{C_3}{2}$$

auf Grund von (8). Für jedes $x\in(0, 1)$ sei $i(x)$ die kleinste positive ganze Zahl mit

$$b_1\Psi_1(x)+\dots+b_{i(x)}\Psi_{i(x)}(x)=\max_{1\leqq m\leqq N}\left(b_1\Psi_1(x)+\dots+b_m\Psi_m(x)\right),$$

und sei $H_i=\{x\in=(0,1): i(x)=i\}$ $(i=1, \dots, N)$. Wir setzen $H_i^*=H_i\setminus G$ $(i=1, \dots, N)$. Offensichtlich sind die Mengen H_i^* einfach, paarweise disjunkt, weiterhin gelten

$$\operatorname{mes} \bigcup_{i=1}^{N} H_i^* \geqq \frac{C_3}{2},$$

$$\left.\begin{aligned} b_1 \Psi_1(x)+\dots+b_i\Psi_i(x) &\geqq \frac{C_2}{2}\|b\|, \\ b_{i+1}\Psi_{i+1}(x)+\dots+b_N\Psi_N(x) &\leqq -\frac{C_2}{2}\|b\| \end{aligned}\right\} \quad (x\in H_i^*)$$

$(i=1, \dots, N)$. Ferner gibt es solche umkehrbar eindeutige, messtreue Transformation T_1 von $(0, 1)$ auf sich selbst derart, dass jede Menge $\bar{H}_i=T_1H_i^*$ $(i=1, \dots, N)$ ein Intervall ist, die Funktionen $\bar{\Psi}_n(x)=\Psi_n(T_1^{-1}x)$ $(n=1, \dots, N)$ Treppenfunktionen sind, und sie ein orthonormiertes System in $(0, 1)$ bilden. Es seien $1\leqq i_1<\dots<i_\varrho\leqq N$ die Indizes i, für die $\operatorname{mes} \bar{H}_i\geqq C_3/4N$ bestehen. Offensichtlich besteht

$$\sum_{r=1}^{\varrho} \operatorname{mes} \bar{H}_{i_r} \geqq \frac{C_3}{4}.$$

Dann gilt

$$\operatorname{mes} \bar{H}_{i_r} = \bar{\alpha}_r\frac{C_3}{8N}+\beta_r\frac{C_3}{8N};$$

wobei $\bar{\alpha}_r$ eine positive ganze Zahl und $0\leqq\beta_r<1$ $(r=1, \dots, \varrho)$ ist. Es sei $\bar{\bar{H}}_{i_r}$ ein Teilintervall von $\bar{H}_{i_r}$ mit

$$\operatorname{mes} \bar{\bar{H}}_{i_r} = \bar{\alpha}_i\frac{C_3}{8N} \quad (r=1, \dots, \varrho).$$

Dann besteht

$$\sum_{r=1}^{\varrho} \operatorname{mes} \bar{\bar{H}}_{i_r} = (\bar{\alpha}_1+\dots+\bar{\alpha}_\varrho)\frac{C_3}{8N} \geqq \frac{C_3}{8},$$

woraus $\bar{\alpha}_1+\dots+\bar{\alpha}_\varrho\geqq N$ folgt. Es seien $\alpha_1, \dots, \alpha_\varrho$ positive ganze Zahlen mit $\alpha_r\leqq\leqq\bar{\alpha}_r$ $(r=1, \dots, \varrho)$ und $\alpha_1+\dots+\alpha_\varrho=N$. Weiterhin sei J_r ein Teilintervall von $\bar{\bar{H}}_{i_r}$ mit $\operatorname{mes} J_r=C_3\alpha_r/8N$ $(r=1, \dots, \varrho)$. Dann gilt

$$\sum_{r=1}^{\varrho} \operatorname{mes} J_r = \frac{C_3}{8}.$$

Wir teilen das Intervall J_r in α_r paarweise disjunkte Teilintervalle $J_r^{(l)}$ $(l=1, \dots, \alpha_r)$ mit

$$\operatorname{mes} J_r^{(l)} = \frac{C_3}{8N} \quad (l=1, \dots; \alpha_r,\ r=1, \dots, \varrho).$$

Die Intervalle $J_r^{(l)}$ ($l=1, \ldots, \alpha_r$; $r=1, \ldots, \varrho$) bezeichnen wir der Reihe nach mit $\bar{I}_1, \ldots, \bar{I}_N$. Offensichtlich gibt es solche umkehrbar eindeutige, messtreue Transformation T_2 von $(0, 1)$ auf sich selbst derart, dass jede Menge $I_i=T_2\bar{I}_i$ ($i=1, \ldots, N$) ein Intervall ist, die Funktionen $\Psi_n(x)=\bar{\Psi}_n(T_2^{-1}x)$ ($n=1, \ldots, N$) Treppenfunktionen sind, und sie ein orthonormiertes System in $(0, 1)$ bilden, weiterhin $\bigcup\limits_{i=1}^{N} I_i=(0, C_3/8)$ erfüllt ist.

Für diese Funktionen $\Psi_n(x)$ ($n=1, \ldots, N$) und für diese Intervalle I_i ($i=1, \ldots, N$) sind alle Forderungen des Hilfssatzes III erfüllt.

Auf Grund der Bedingungen des Satzes B gibt es eine ganze Zahl σ (0 oder 1), für die

$$\sum_{\nu=1}^{\infty} \sqrt{\sum_{n=N_{2\nu+\sigma-1}+1}^{N_{2\nu+\sigma}} a_n^2 \log^2 n} = \infty \tag{9}$$

ist. Weiterhin sei ν_0 eine positive ganze Zahl mit

$$1 \leqq \frac{C_1^2 C_2^2 C_3}{16} \log^2 \frac{2^{2^{2\nu+\sigma-1}}}{2} \quad (\nu \geqq \nu_0). \tag{10}$$

Im Folgenden bezeichnet $r_n(x)$ die n-te Rademachersche Funktion: $r_n(x)= =\operatorname{sign} \sin 2^n \pi x$ ($n=0, 1, \ldots$).

Hilfssatz IV. *Es seien ν_1, ν_2 ganze Zahlen mit $\nu_0 \leqq \nu_1 \leqq \nu_2$, und $\{a_n\}_1^\infty$ eine Folge mit (5). Dann gibt es ein orthonormiertes System $\{\Phi_n(x)\}_{N_{2\nu_1+\sigma-1}+1}^{N_{2\nu_2+\sigma+1}}$ der Treppenfunktionen in $(0, 4)$ und eine Folge der paarweise disjunkten, einfachen Mengen $I_i(\nu_2)$ $(\subseteqq(0, 1))$ ($i=1, \ldots, 2^{2^{2\nu_2+\sigma}}$) mit den folgenden Eigenschaften:*

Es ist

$$\operatorname{mes} I_i(\nu_2) = \frac{1}{2^{2^{2\nu_2+\sigma}}} \quad (i = 1, \ldots, 2^{2^{2\nu_2+\sigma}}). \tag{11}$$

Es gilt

$$\Phi_n(x) = \begin{cases} r_n(x), & x\in(3, 4), \\ 0, & \text{sonst} \end{cases} \quad (N_{2\nu+\sigma} < n \leqq N_{2\nu+\sigma+1};\ \nu = \nu_1, \ldots, \nu_2). \tag{12}$$

Es gibt nichtleere Intervalle $J_1(\nu_2)$ $(\subseteqq(1, 2))$, $J_2(\nu_2)$ $(\subseteqq(2, 3))$ derart, dass

$$\Phi_n(x) = 0 \quad \left(x\in J_1(\nu_2) \cup J_2(\nu_2);\ n = N_{2\nu_1+\sigma-1}+1, \ldots, N_{2\nu_2+\sigma+1}\right) \tag{13}$$

gilt. Weiterhin besitzt die Summe

$$\sum_{n=N_{2\nu_1+\sigma-1}+1}^{N_{2\nu_2+\sigma+1}} a_n \Phi_n(x)$$

eine Anordnung

$$\sum_{k=N_{2\nu_1+\sigma-1}+1}^{N_{2\nu_2+\sigma+1}} a_{n_k(\nu_2)}\Phi_{n_k(\nu_2)}(x)$$

ihrer Glieder derart, dass mit gewissen Indizes $m_1(i, \nu_2)$, $m_2(i, \nu_2)$ $(N_{2\nu_1+\sigma-1} < m_1(i, \nu_2) \leqq m_2(i, \nu_2) \leqq N_{2\nu_2+\sigma+1})$

(14)

$$\sum_{k=m_1(i,\nu_2)}^{m_2(i,\nu_2)} a_{n_k(\nu_2)}\Phi_{n_k(\nu_2)}(x) \geqq \frac{C_1C_2\sqrt{C_3}}{\sqrt{128}} \sum_{\nu=\nu_1}^{\nu_2} \sqrt{\sum_{n=N_{2\nu+\sigma-1}+1}^{N_{2\nu+\sigma}} a_n^2 \log^2 n} \quad (x\in I_i(\nu_2))$$

$(i = 1, \dots, 2^{2^{2\nu_2+\sigma}})$ *besteht.*

Beweis des Hilfssatzes IV. Wir können den Hilfssatz III für die Folge $b=\{a_{N_{2\nu_1+\sigma-1}+1}, \dots, a_{N_{2\nu_1+\sigma}}, 0, \dots\}$ anwenden; nämlich wegen (10) ist (8) offensichtlich erfüllt. Die sich ergebenden Funktionen, bzw. Intervalle bezeichnen wir mit $\Psi_n(x)$ $(n=N_{2\nu_1+\sigma-1}+1, \dots, N_{2\nu_1+\sigma})$, bzw. mit I_i $(i=1, \dots, 2^{2^{2\nu_1+\sigma}})$. Wir setzen

$$\Phi_n(x) = \begin{cases} \sqrt{\frac{C_3}{8}}\,\Psi_n\left(\frac{C_3}{8}x\right), & x\in(0, 1),\\ \Psi_n(x-1), & x\in(1+C_3/8, 2),\\ 0, & \text{sonst}\end{cases} \quad (n = N_{2\nu_1+\sigma-1}+1, \dots, N_{2\nu_1+\sigma}),$$

$$\Phi_n(x) = \begin{cases} r_n(x), & x\in(3, 4),\\ 0, & \text{sonst}\end{cases}$$

$(n=N_{2\nu_1+\sigma}+1, \dots, N_{2\nu_1+\sigma+1})$, $J_1(\nu_1)=(1, 1+C_3/8)$, $J_2(\nu_1)=(2, 3)$. Weiterhin sei $I_i(\nu_1)$ $(i=1, \dots, 2^{2^{2\nu_1+\sigma}})$, dasjenige Intervall, welches aus I_i mit der Transformation $y=8/C_3\cdot x$ entsteht. Endlich sei $n_k(\nu_1)=k$ $(k=N_{2\nu_1+\sigma-1}+1, \dots, N_{2\nu_1+\sigma+1})$. Auf Grund der Hilfssätze I, III sind (11)—(14) für das System $\{\Phi_n(x)\}_{N_{2\nu_1+\sigma-1}+1}^{N_{2\nu_1+\sigma+1}}$, für die Mengen $I_i(\nu_1)$ $(i=1, \dots, 2^{2^{2\nu_1+\sigma}})$, für die Intervalle $J_1(\nu_1)$, $J_2(\nu_1)$ und für die Anordnung $n_k(\nu_1)$ $(k=N_{2\nu_1+\sigma-1}+1, \dots, N_{2\nu_1+\sigma+1})$ im Falle $\nu_2=\nu_1$ erfüllt.

Wenn $\nu_2>\nu_1$ ist, dann sei ν^* eine ganze Zahl mit $\nu_1\leqq\nu^*<\nu_2$. Wir nehmen an, dass das orthonormierte System $\{\Phi_n(x)\}_{N_{2\nu_1+\sigma-1}+1}^{N_{2\nu^*+\sigma+1}}$ der Treppenfunktionen in $(0, 4)$, die paarweise disjunkten, einfachen Mengen $I_i(\nu^*)$ $(i=1, 2\dots, 2^{2^{2\nu^*+\sigma}})$ und die Anordnung $n_k(\nu^*)$ $(k=N_{2\nu_1+\sigma-1}+1, \dots, N_{2\nu^*+\sigma+1})$ derart definiert sind, dass (11)—(14) mit gewissen nichtleeren Intervallen $J_1(\nu^*)$ $(\subseteqq(1, 2))$, $J_2(\nu^*)$ $(\subseteqq(2, 3))$ und mit gewissen Indizes $m_1(i, \nu^*)$, $m_2(i, \nu^*)$, $(N_{2\nu_1+\sigma-1}<m_1(i, \nu^*)\leqq m_2(i, \nu^*)\leqq N_{2\nu^*+\sigma+1})$ im Falle $\nu_2=\nu^*$ erfüllt sind.

Füı jede ganze Zahl i, $1 \leqq i \leqq 2^{2^{2\nu^*+\sigma}}$ setzen wir die Indexmengen

$$Z_i^{(1)} = \{a+(i-1)c_0+1, \dots, a+(i-1)c_0+c_0/2\} \cup$$

$$\cup \left(\bigcup_{s=1}^{2^{2\nu^*+\sigma+1}-1} \{a+2^s b+(i-1)c_s+1, \dots, a+2^s b+(i-1)c_s+c_s/2\} \right),$$

$$Z_i^{(2)} = \{a+(i-1)c_0+c_0/2+1, \dots, a+ic_0\} \cup$$

$$\cup \left(\bigcup_{s=1}^{2^{2\nu^*+\sigma+1}-1} \{a+2^s b+(i-1)c_s+c_s/2+1, \dots, a+2^s b+ic_s\} \right),$$

$$Z_i = \{a+(i-1)c_0+1, \dots, a+ic_0\} \cup$$

$$\cup \left(\bigcup_{s=1}^{2^{2\nu^*+\sigma+1}-1} \{a+2^s b+(i-1)c_s+1, \dots, a+2^s b+ic_s\} \right),$$

wobei $a = N_{2\nu^*+\sigma+1}$, $b = 2^{2^{2\nu^*+\sigma+1}}$, $c_0 = 2^{2^{2\nu^*+\sigma+1}+1}/2^{2^{2\nu^*+\sigma}}$, $c_s = 2^{2^{2\nu^*+\sigma+1}+s}/2^{2^{2\nu^*+\sigma}}$ $(s=1, \dots, 2^{2\nu^*+\sigma+1}-1)$ ist. Offensichtlich sind die Mengen $Z_i^{(1)}, Z_i^{(2)}$ $(i=1, \dots$ $\dots, 2^{2^{2\nu^*+\sigma}})$ paarweise disjunkt, es gelten

$$Z_i^{(1)} \cup Z_i^{(2)} = Z_i \quad (i=1, \dots, 2^{2^{2\nu^*+\sigma}}), \quad \bigcup_{i=1}^{2^{2^{2\nu^*+\sigma}}} Z_i = \{N_{2\nu^*+\sigma+1}+1, \dots, N_{2(\nu^*+1)+\sigma}\},$$

die Mächtigkeiten der Mengen $Z_i^{(1)}, Z_i^{(2)}$ sind gleich mit $2^{2^{2(\nu^*+1)+\sigma}}/2 \cdot 2^{2^{2\nu^*+\sigma}} = M_0$. Die Elemente von $Z_i^{(1)}$ bezeichnen wir in natürlicher Anordnung mit $l_1(i), \dots, l_{M_0}(i)$, weiterhin bezeichnen wir die Elemente von $Z_i^{(2)}$ in natürlicher Anordnung mit $l_{M_0+1}(i), \dots, l_{2M_0}(i)$ $(i=1, \dots, 2^{2^{2\nu^*+\sigma}})$. Auf Grund von (5) sind die Folgen $\{a_{l_1(i)}, \dots, a_{l_{M_0}(i)}\}$, $\{a_{l_{M_0+1}(i)}, \dots, a_{l_{2M_0}(i)}\}$ mit dieselber Folge gleich; diese Folge bezeichnen wir mit $b^{(0)} = \{b_1^{(0)}, \dots, b_{M_0}^{(0)}, 0, \dots\}$.

Für jede ganze Zahl i $(1 \leqq i \leqq 2^{2^{2\nu^*+\sigma}})$ werden wir die Funktionen $\Phi_{l_j(i)}(x)$ $(j=1, \dots, 2M_0)$ definieren. Es seien $I_1^*(i)$ $(\subseteqq J_1(\nu^*))$, $I_2^*(i)$ $(\subseteqq J_2(\nu^*))$ $(i=1, \dots$ $\dots, 2^{2^{2\nu^*+\sigma}})$ paarweise disjunkte, nichtleere Intervalle mit

$$J_1^*(\nu^*+1) = J_1(\nu^*) \setminus \bigcup_{i=1}^{2^{2^{2\nu^*+\sigma}}} I_1^*(i) \neq \emptyset, \quad J_2(\nu^*+1) = J_2(\nu^*) \setminus \bigcup_{i=1}^{2^{2^{2\nu^*+\sigma}}} I_2^*(i) \neq \emptyset.$$

Wir wenden den Hilfssatz III auf die Folge $b^{(0)}$ an; wegen (10) besteht (8) offensichtlich, und so kann man den Hilfssatz III anwenden. Die sich ergebenden Funktionen, bzw. Intervalle bezeichnen wir mit $\Psi_n(x)$ $(n=1, \dots, M_0)$ bzw. mit I_j $(j=1, \dots$

..., M_0). Wir setzen

$$\Psi_n^{(1)}(x) = \begin{cases} \sqrt{\dfrac{C_3}{8}}\,\Psi_n\!\left(\dfrac{C_3}{8}x\right), & (x\in(0,1), \\ 0, & \text{sonst}, \end{cases}$$

$$\Psi_n^{(2)}(x) = \begin{cases} \sqrt{\dfrac{8-C_3}{8}}\,\Psi_n\!\left(\left(1-\dfrac{C_3}{8}\right)x+\dfrac{C_3}{8}\right), & (x\in(0,1), \\ 0, & \text{sonst} \end{cases}$$

($n=1, \ldots, M_0$). Weiterhin sei $\bar{I}_j$ ($j=1, \ldots, M_0$) das Intervall, welches aus I_j mit der Transformation $y=\dfrac{C_3}{8}x$ entsteht.

Für eine in $(0, 1)$ definierte Funktion $f(x)$ und für ein Intervall $I=(\alpha, \beta)$ $(\subseteqq(0, 1))$ sei

$$f(I; x) = \begin{cases} f\!\left(\dfrac{x-\alpha}{\beta-\alpha}\right), & \alpha < x < \beta, \\ 0, & \text{sonst}, \end{cases}$$

weiterhin, für eine Menge H $(\subseteqq(0, 1))$ sei $H(I)$ die Menge, die aus H mit der Transformation $y=(\beta-\alpha)x+\alpha$ entsteht.

Da die Funktionen $\Phi_n(x)$ ($n=N_{2\nu_1+\sigma-1}+1, \ldots, N_{2\nu^*+\sigma+1}$) Treppenfunktionen sind, gibt es eine Einteilung von $I_i(\nu^*)$ in paarweise disjunkte Intervalle $J_r(i)$ ($r=1, \ldots, \varrho$) derart, dass jede Funktion $\Phi_n(x)$ ($n=N_{2\nu_1+\sigma-1}+1, \ldots, N_{2\nu^*+\sigma+1}$) in jedem Intervall $J_r(i)$ ($r=1, \ldots, \varrho$) konstant ist; die zwei Hälfte von $J_r(i)$ bezeichnen wir mit $J_r'(i)$, bzw. mit $J_r''(i)$ ($r=1, \ldots, \varrho$). Wir setzen

$$\Psi_n(i; x) =$$

$$= \frac{1}{\sqrt{\operatorname{mes} I_i(\nu^*)}}\left(\sum_{r=1}^{\varrho}\Psi_n^{(1)}(J_r'(i); x)-\sum_{r=1}^{\varrho}\Psi_n^{(1)}(J_r''(i); x)\right)+\frac{1}{\sqrt{\operatorname{mes} I_1^*(i)}}\,\Psi_n^{(2)}(I_1^*(i); x)$$

($n=1, \ldots, M_0$), und

$$I_{(i-1)2M_0+j}(\nu^*+1) = \bigcup_{r=1}^{\varrho}\bar{I}_j(J_r'),$$

$$I_{(i-1)2M_0+M_0+j}(\nu^*+1) = \bigcup_{r=1}^{\varrho}\bar{I}_j(J_r'') \quad (j=1, \ldots, M_0).$$

Auf Grund des Hilfssatzes III und der Definition ist es offensichtlich, dass die Mengen $I_l(\nu^*+1)$ $(\subseteqq(0, 1))$ ($l=1, \ldots, 2^{2^{2(\nu^*+1)+\sigma}}$) paarweise disjunkt und einfach sind, weiterhin (11) für $\nu_2=\nu^*+1$ besteht. Wir setzen

$$\overline{\Psi}_n(i;x)=\begin{cases}\frac{1}{\sqrt{2}}\Psi_n(i;x), & x\in(0,2),\\ 0, & x\in(0,4)\setminus I_2^*(i)\end{cases}\qquad (n=1,\dots,M_0).$$

$$\overline{\Psi}_n(i;x)=\begin{cases}\frac{1}{\sqrt{2}}\Psi_n(i;x), & x\in(0,2),\\ 0, & x\in(0,4)\setminus I_2^*(i)\end{cases}\qquad (n=M_0+1,\dots,2M_0).$$

Man kann die Funktionen $\overline{\Psi}_n(i;x)$ $(n=1,\dots,2M_0)$ in $I_2^*(i)$ leicht derart definieren, dass sie in $I_2^*(i)$ Treppenfunktionen sind, und ein orthonormiertes System in $(0,4)$ bilden. Wir setzen

$$\Phi_{l_j(i)}(x)=\overline{\Psi}_j(i;x)\quad (j=1,\dots,2M_0;\ i=1,\dots,2^{2^{2\nu^*+\sigma}}).$$

Auf Grund des Hilfssatzes III und der Definition ist es offensichtlich, dass diese Funktionen in $(0,4)$ Treppenfunktionen sind, und die Funktionen $\Phi_n(x)$ $(n=N_{2\nu_1+\sigma-1}+1,\dots$ $\dots,N_{2(\nu^*+1)+\sigma})$ ein orthonormiertes System in $(0,4)$ bilden, weiterhin (13) für $\nu_2=\nu^*+1$ gilt. Auf Grund der Hilfssätze I, III und der Definition folgt durch einfache Rechnung, dass mit gewissen Indizes $\mu_j(i)$ $(1\leqq\mu_j(i)\leqq M_0;\ j=1,\dots,M_0)$

$$(15)\qquad \begin{aligned}&a_{l_{M_0+1}(i)}\Phi_{l_{M_0+1}(i)}(x)+\dots+a_{l_{M_0+\mu_j(i)}(i)}\Phi_{l_{M_0+\mu_j(i)}(i)}(x)\geqq\\ &\geqq\frac{C_1C_2\sqrt{C_3}}{\sqrt{128}}\sqrt{\sum_{n=N_{2\nu^*+\sigma+1}+1}^{N_{2(\nu^*+1)+\sigma}}a_n^2\log^2 n}\quad (x\in I_{(i-1)2M_0+j}),\\ &a_{l_{\mu_j(i)}(i)}\Phi_{l_{\mu_j(i)}(i)}(x)+\dots+a_{i_{M_0}(i)}\Phi_{l_{M_0}(i)}(x)\geqq\\ &\geqq\frac{C_1C_2\sqrt{C_3}}{\sqrt{128}}\sqrt{\sum_{n=N_{2\nu^*+\sigma+1}+1}^{N_{2(\nu^*+1)+\sigma}}a_n^2\log^2 n}\quad (x\in I_{(i-1)2M_0+M_0+j})\end{aligned}$$

$(j=1,\dots,M_0;\ i=1,\dots,2^{2^{2\nu^*+\sigma}})$ gelten. Wir setzen endlich

$$\Phi_n(x)=\begin{cases}r_n(x), & x\in(3,4),\\ 0, & \text{sonst}\end{cases}$$

$(n=N_{2(\nu^*+1)+\sigma}+1,\dots,N_{2(\nu^*+1)+\sigma+1})$. Offensichtlich sind die Funktionen $\Phi_n(x)$ $(n=N_{2\nu_1+\sigma-1}+1,\dots,N_{2(\nu^*+1)+\sigma+1})$ Treppenfunktionen und bilden ein orthonormiertes System in $(0,4)$, weiterhin gilt (12) für $\nu_2=\nu^*+1$.

Für jede ganze Zahl i, $0\leqq i<2^{2^{2\nu^*+\sigma}}$, definieren wir eine Anordnung $n_k(i,\nu^*)$ $\big(k=N_{2\nu_1+\sigma-1}+1,\dots,N_{2\nu^*+\sigma+1}+2M_0(i+1)\big)$ der Indizes

$$n\in\{N_{2\nu_1+\sigma-1}+1,\dots,N_{2\nu^*+\sigma+1}\}\cup\left(\bigcup_{j=1}^{i+1}Z_j\right).$$

Es sei $n_k(0, \nu^*)=n_k(\nu^*)$ $(k=N_{2\nu_1+\sigma-1}+1, \dots, N_{2\nu^*+\sigma+1})$, und

$$n_k(i+1, \nu^*) = \begin{cases} n_k(i, \nu^*), & k = N_{2\nu_1+\sigma-1}+1, \dots, m_1(i+1, \nu^*)-1, \\ m_1(i+1, \nu^*)-1+l_{k-m_1(i+1,\nu^*)+1}(i+1), & \\ \qquad k = m_1(i+1, \nu^*), \dots, m_1(i+1; \nu^*)+M_0-1, & \\ n_{k-M_0}(i, \nu^*), & k = m_1(i+1, \nu^*)+M_0, \dots, m_2(i+1; \nu^*)+M_0; \\ m_2(i+1, \nu^*)+l_{k-m_2(i+1,\nu^*)}(i+1), & \\ \qquad k = m_2(i+1, \nu^*)+M_0+1, \dots, m_2(i+1, \nu^*)+2M_0, & \\ n_{k-2M_0}(i, \nu^*), & k = m_2(i+1, \nu^*)+2M_0+1, \dots, N_{2\nu^*+\sigma+1}+2M_0(i+1) \end{cases}$$

$(i=1, \dots, 2^{2^{2\nu^*+\sigma}}-1)$. Wir setzen

$$n_k(\nu^*+1) = \begin{cases} n_k(2^{2^{2\nu^*+\sigma}}, \nu^*), & k = N_{2\nu_1+\sigma-1}+1, \dots, N_{2(\nu^*+1)+\sigma}, \\ k, & k = N_{2(\nu^*+1)+\sigma}+1, \dots, N_{2(\nu^*+1)+\sigma+1}. \end{cases}$$

Auf Grund der Voraussetzung, der Definitionen und der Ungleichungen (15) kann man leicht sehen, dass (14) für die Funktionen $\Phi_n(x)$ $(n=N_{2\nu_1+\sigma-1}+1, \dots, N_{2(\nu^*+1)+\sigma+1})$, für die Mengen $I_i(\nu^*+1)$ $(i=1, \dots, 2^{2^{2(\nu^*+1)+\sigma}})$ und für die Anordnung $n_k^*(\nu^*+1)$ $(k=N_{2\nu_1+\sigma-1}+1, \dots, N_{2(\nu^*+1)+\sigma+1})$ mit gewissen Indizes $m_1(i, \nu^*+1), m_2(i, \nu^*+1)$ $\left(N_{2\nu_1+\sigma-1}<m_1(i, \nu^*+1)\leqq m_2(i, \nu^*+1)\leqq N_{2(\nu^*+1)+\sigma+1}\right)$ im Falle $\nu_2=\nu^*+1$ erfüllt ist.

Den Hilfssatz IV bekommen wir dann durch Induktion.

3. Beweis des Satzes B. Aus (9) folgt, dass eine Indexfolge $\{\bar\nu_s\}_1^\infty$ mit $\nu_0\leqq\bar\nu_1<\dots<\bar\nu_s<\dots$ und

$$\sum_{\nu=\bar\nu_s+1}^{\bar\nu_{s+1}} \frac{C_1C_2\sqrt{C_3}}{\sqrt{128}} \sqrt{\sum_{n=N_{2\nu+\sigma-1}}^{N_{2\nu+\sigma}} a_n^2\log^2 n} \geqq 2s \quad (s = 1, 2, \dots) \tag{16}$$

existiert. Für jede ganze Zahl s $(s=1, 2, \dots)$ wenden wir den Hilfssatz IV im Falle $\nu_1=\bar\nu_s+1$, $\nu_2=\bar\nu_{s+1}$ an. Die sich ergebenden Funktionen, bzw. die sich ergebende Anordnung bezeichnen wir mit $\Phi_n(s; x)$ $(n=N_{2(\bar\nu_s+1)+\sigma-1}+1, \dots, N_{2\bar\nu_{s+1}+\sigma+1})$, bzw. mit $\bar n_k(s)$ $(k=N_{2(\bar\nu_s+1)+\sigma-1}+1, \dots, N_{2\bar\nu_{s+1}+\sigma+1})$. Auf Grund des Hilfssatzes IV sind diese Funktionen Treppenfunktionen, und sie bilden ein orthonormiertes System in $(0, 4)$, weiterhin auf Grund von (16) gilt

$$\max_{N_{2(\bar\nu_s+1)+\sigma-1}<i\leqq j\leqq N_{2\bar\nu_{s+1}+\sigma+1}} \left|\sum_{k=i}^{j} a_{\bar n_k(s)}\Phi_{\bar n_k(s)}(s; x)\right| \geqq 2s \quad (x\in(0, 1)) \tag{17}$$

$(s = 1, 2, \dots)$. Wir setzen

$$\bar\Phi_n(x) = \frac{1}{2}\Phi_n\left(s; \frac{x}{4}\right) \quad \left(x\in(0, 1);\ n = N_{2(\bar\nu_s+1)+\sigma-1}+1, \dots, N_{2\bar\nu_{s+1}+\sigma+1}\right).$$

Aus (17) folgt

$$\max_{N_{2(\bar{\nu}_s+1)+\sigma-1}<i\leqq j\leqq N_{2\bar{\nu}_{s+1}+\sigma+1}}\left|\sum_{k=i}^{j} a_{\bar{n}_k(s)}\bar{\Phi}_{\bar{n}_k(x)}(x)\right|\geqq s\quad\left(x\in\left(0,\frac{1}{4}\right)=H\right)\tag{18}$$

$(s=1,2,\ldots)$.

Es sei $\varphi_n(x)=r_n(x)$ $(n=1,\ldots,N_{2(\bar{\nu}_1+1)+\sigma-1})$. Es sei s_0 eine nichtnegative ganze Zahl. Wir nehmen an, dass die Funktionen $\varphi_n(x)$ $(n=1,\ldots,N_{2\bar{\nu}_{s_0+1}+\sigma+1})$ und die Mengen $E_1,\ldots,E_{s_0}$ $(\subseteqq(0,1))$ schon derart definiert sind, dass diese Funktionen Treppenfunktionen sind, sie ein orthonormiertes System in $(0,1)$ bilden, diese Mengen einfach und stochastisch unabhängig sind, weiterhin

$$\operatorname{mes} E_s=1/4,\tag{19}$$

und

$$\max_{N_{2(\bar{\nu}_s+1)+\sigma-1}<i\leqq j\leqq N_{2\bar{\nu}_{s+1}+\sigma+1}}\left|\sum_{k=i}^{j} a_{\bar{n}_k(s)}\varphi_{\bar{n}_k(s)}(x)\right|\geqq s\quad(x\in E_s)\tag{20}$$

für $s=1,\ldots,s_0$ erfüllt sind. Da die Funktionen $\varphi_n(x)$ $(n=1,\ldots,N_{2\bar{\nu}_{s_0+1}+\sigma+1})$ Treppenfunktionen und die Mengen $E_1,\ldots,E_{s_0}$ einfach sind, gibt es eine Einteilung von $(0,1)$ in paarweise disjunkte Intervalle $J_1,\ldots,J_\varrho$ derart, dass jede Funktion $\varphi_n(x)$ $(n=1,\ldots,N_{2\bar{\nu}_{s_0+1}+\sigma+1})$ in jedem Intervall J_r $(r=1,\ldots,\varrho)$ konstant ist, und jede Menge E_s $(s=1,\ldots,s_0)$ die Vereinigung gewisser J_r ist. Die zwei Hälfte von J_r bezeichnen wir mit J'_r, bzw. mit J''_r $(r=1,\ldots,\varrho)$. Wir setzen

$$\varphi_n(x)=\sum_{r=1}^{\varrho}\bar{\Phi}_n(J'_r;\,x)-\sum_{r=1}^{\varrho}\bar{\Phi}_n(J''_r;\,x)\quad(n=N_{2\bar{\nu}_{s_0+1}+\sigma-1}+1,\ldots,N_{2\bar{\nu}_{s_0+2}+\sigma+1}),$$

$$E_{s_0+1}=\left(\bigcup_{r=1}^{\varrho}H(J'_r)\right)\cup\left(\bigcup_{r=1}^{\varrho}H(J''_r)\right).$$

Nach Obigen und nach der Definitionen sind die Funktionen $\varphi_n(x)$ $(n=N_{2\bar{\nu}_{s_0+1}+\sigma-1}+1,\ldots,N_{2\bar{\nu}_{s_0+2}+\sigma+1})$ Treppenfunktionen, die Menge E_{s_0+1} ist einfach, die Funktionen $\varphi_n(x)$ $(n=1,\ldots,N_{2\bar{\nu}_{s_0+2}+\sigma+1})$ bilden ein orthonormiertes System in $(0,1)$, die Mengen $E_1,\ldots,E_{s_0+1}$ sind stochastisch unabhängig, für $s=s_0+1$ gilt (19), weiterhin aus (18) sich ergibt, dass (20) für $s=s_0+1$ auch erfüllt wird. Durch Induktion erhalten wir ein orthonormiertes System $\{\varphi_n(x)\}_1^\infty$ der Treppenfunktionen in $(0,1)$, und eine Folge der einfachen, und stochastisch unabhängigen Mengen E_s $(\subseteqq(0,1))$ $(s=1,2,\ldots)$ derart, dass (19) und (20) für jedes s $(=1,2,\ldots)$ erfüllt werden.

Es seien $n_k=k$ $(k=1,\ldots,N_{2(\bar{\nu}_1+1)+\sigma-1})$, und $n_k=\bar{n}_k(s)$ $(N_{2(\bar{\nu}_s+1)+\sigma-1}<k\leqq$ $\leqq N_{2\bar{\nu}_{s+1}+\sigma+1};\ s=1,2,\ldots)$. Dann gilt

$$\max_{N_{2(\bar{\nu}_s+1)+\sigma-1}<i\leqq j\leqq N_{2\bar{\nu}_{s+1}+\sigma+1}}\left|\sum_{k=i}^{j} a_{n_k}\varphi_{n_k}(x)\right|\geqq s\quad(x\in E_s)\tag{21}$$

für jedes $s=1, 2, \ldots,$ auf Grund von (20). Da die Mengen E_s $(s=1, 2, \ldots)$ stochastisch unabhängig sind, aus (19) folgt mes $\overline{\lim\limits_{s\to\infty}} E_s=1$. Weiterhin aus (21) bekommen wir

$$\overline{\lim_{i,j\to\infty}} \left|\sum_{k=i}^{j} a_{n_k}\varphi_{n_k}(x)\right| = \infty \quad (x\in \overline{\lim_{s\to\infty}} E_s).$$

Damit haben wir Satz B vollständig bewiesen.

Schriftenverzeichnis

[1] K. Tandori, Über die orthogonalen Funktionen. X (Unbedingte Konvergenz), *Acta Sci. Math.*, **23** (1962), 185—221.

[2] K. Tandori, Über die Konvergenz der Orthogonalreihen. II, *Acta Sci. Math.*, **25** (1964), 219—232.

BOLYAI INSTITUT
UNIVERSITÄT SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, UNGARN

Acta Sci. Math., 47 (1984), 201—204

Remarks to a paper of V. Komornik

I. JOÓ

Let G be an arbitrary open interval on the real line, $q_1, q_2, \ldots, q_n \in L_1^{\mathrm{loc}}(G)$ arbitrary complex functions, and consider the formal differential operator

$$lu = u^{(n)} + q_1 u^{(n-1)} + \ldots + q_n u. \tag{1}$$

Let λ be an arbitrary complex number. The function $u \equiv 0$ is called an eigenfunction of order -1 with eigenvalue λ of the operator l. Assume that the eigenfunctions of order $m-1 \geqq -1$ are already defined, then a function $u\colon G \to \mathbf{C}$, $u \not\equiv 0$ is called an eingenfunction of order m with eigenvalue λ of the operator l, if the functions $u, u', \ldots, u^{(n-1)}$ are locally absolutely continuous on G and there exists an eigenfunction u^* of order $m-1$ of the operator l with the same eingenvalue such that a.e. on G

$$lu = \lambda u + u^*. \tag{2}$$

The eigenfunctions of higher order play important role in the theory of expansions [2]. Connected with this problem an upper estimate was given for the sup-norm of the eigenfunctions in [1], on the basis of Titchmarsh formula. It was shown in [3] that this result is exact from the point of view of dependence on the eigenvalue. Later on V. Komornik [4] generalized the Titchmarsh formula for the differential operator (1) and using this result he obtained upper estimates for the eigenfunctions of (1), too.

The present paper has two purposes. First we give a new proof for Komornik's formula which is simpler than the former one and provides the coefficients in explicit form. Furthermore this approach sheds light on the inner beauty of this formula. As an application of this explicit expression we can prove exact lower estimates for the eigenfunctions of the operator (1).

For the sake of simplicity we consider only the operator

$$lu = u^{(n)}, \quad G = \mathbf{R}. \tag{3}$$

Received November 16, 1982.

The general case hence follows by the ideas of the paper [4] (by the variation of the constants).

1. Given a complex number $\lambda=\mu^n$ we denote by $S_k(\mu)$ the elementary symmetric polynomial of degree k of the variables $e^{\mu\omega_1}, e^{\mu\omega_2}, \dots, e^{\mu\omega_n}$ (where $\omega_1, \dots, \omega_n$ denotes the n-th roots of the unity) with the main coefficient $(-1)^k$, and

$$f_k(\mu) = (-1)^k S_{n-k}(e^{\mu\omega_1}, \dots, e^{\mu\omega_n}).$$

Obviously $f_k(\mu)=f_k(\mu\omega_1)=\dots=f_k(\mu\omega_n)$. Introduce also the functions $f_{0,k} \overset{\text{def}}{=} f_k$ $(k=0, 1, \dots, n)$,

$$f_{m,k} \overset{\text{def}}{=} \sum_{r=\max(0,k-mn)}^{\min(k,n)} f_r \cdot f_{m-1,k-r} \quad (m = 1, 2, \dots;\ k = 0, 1, \dots, (m+1)n).$$

Theorem 1. *Let u be an arbitrary eigenfunction of order $\leqq m$ of the operator* (3) *with some eigenvalue $\lambda=\mu^n$ $(m=0, 1, \dots)$. Then for any $x, t \in \mathbf{R}$,*

$$\sum_{k=0}^{(m+1)n} f_{m,k}(\mu t) \cdot u(x+kt) = 0. \tag{4}$$

Proof. First we show that

$$\hat{u}(x) \overset{\text{def}}{=} \sum_{k=0}^{n} f_k(\mu t) \cdot u(x+kt) \tag{5}$$

is an eigenfunction of order $\leqq m-1$ of the operator (3) with the eigenvalue λ. We consider only the case $\lambda \neq 0$ (the case $\lambda=0$ is similar). Then u has the form

$$u(x) = \sum_{p=1}^{n} \sum_{r=0}^{m} a_{pr} x^r e^{\mu\omega_p x}$$

with some constants a_{pr}. Then

$$\hat{u}(x) = \sum_{k=0}^{n} f_k(\mu t) \sum_{p=1}^{n} \sum_{r=0}^{m} a_{pr}(x+kt)^r e^{\mu\omega_p(x+kt)} =$$

$$= \sum_{k=0}^{n} f_k(\mu t) \sum_{p=1}^{n} \sum_{r=0}^{m} a_{pr} \sum_{s=0}^{r} \binom{r}{s} x^s (kt)^{r-s} e^{\mu\omega_p x} e^{k\mu\omega_p t} =$$

$$= \sum_{p=1}^{n} \sum_{s=0}^{m} \left[\sum_{r=s}^{m} a_{pr} \binom{r}{s} \sum_{k=0}^{n} (kt)^{r-s} f_k(\mu t) e^{k\mu\omega_p t} \right] x^s e^{\mu\omega_p x} = \sum_{p=1}^{n} \sum_{s=0}^{m} b_{ps} x^s e^{\mu\omega_p x},$$

where the numbers b_{ps} do not depend on x but depend on t. It suffices to show that $b_{1m}=\dots=b_{nm}=0$. But for any $1 \leqq p \leqq n$,

$$b_{pm} = a_{pm} \prod_{q=1}^{n} (e^{\mu\omega_q t} - e^{\mu\omega_p t}) = 0.$$

Now we prove the formula (4) by induction on m. For $m=0$ it follows directly from the result just proved because the eigenfunctions of order -1 are identically zero. Assume (4) is valid for $m-1\geqq 0$. Then it is true also for m. Indeed, applying the induction hypothesis for $\hat{u}$ defined in (5) we obtain:

$$0=\sum_{k=0}^{mn} f_{m-1,k}(\mu t)\cdot\hat{u}(x+kt)=\sum_{k=0}^{mn} f_{m-1,k}(\mu t)\sum_{l=0}^{n} f_l(\mu t)u(x+kt+lt)=$$

$$=\sum_{r=0}^{(m+1)n}\left[\sum_{s=\max(0,r-mn)}^{\min(r,n)} f_s\cdot f_{m-1,r-s}(\mu t)\right]u(x+rt)=\sum_{r=0}^{(m+1)n} f_{m,r}(\mu t)u(x+rt).$$

Theorem 1 is proved.

2. We prove the following result.

Theorem 2. *Given any compact interval $K=[a,b]\subset\mathbf{R}$ there exists a positive constant $C=C(m)$ such that for any eigenfunction of order $m\geqq 0$ of the operator* (3) *with the eigenvalue λ*

$$\|u\|_{L^\infty(K)}\leqq C(1+|\operatorname{Re}\mu|)^{1/p}\|u\|_{L^p(K)}\quad(1\leqq p\leqq\infty). \tag{6}$$

Here μ denotes such an n-th root of λ for which $|\operatorname{Re}\mu|$ is minimal.

Proof. Given any $\lambda=\mu^n\in\mathbf{C}$ denote by $\mu_1,\dots,\mu_n$ the n-th roots of λ such that $\operatorname{Re}\mu_1\geqq\dots\geqq\operatorname{Re}\mu_n$. Obviously, putting $m=[n/2]$ or $m=[n/2]+1$, we have $\operatorname{Re}\mu_m\geqq 0$ and $\operatorname{Re}\mu_{m+1}\leqq 0$. We distinguish two cases: $|\operatorname{Re}\mu_m|\leqq|\operatorname{Re}\mu_{m+1}|$ or $|\operatorname{Re}\mu_m|>|\operatorname{Re}\mu_{m+1}|$ (the second case can occur only if n is odd). Let us consider first the case $|\operatorname{Re}\mu_m|\leqq|\operatorname{Re}\mu_{m+1}|$. Then obviously for any $t>0$

$$\left|\frac{f_k(\mu t)}{e^{(\mu_1+\dots+\mu_{m-1})t}}\right|\leqq C\quad\text{if}\quad 0\leqq k\leqq n,\ n-k\neq m.$$

and

$$\left|\frac{f_{n-m}(\mu t)}{e^{(\mu_1+\dots+\mu_{m-1})t}}-e^{\mu_m t}\right|\leqq C$$

where C is an absolute constant independent of μ and t. Hence, dividing the formula (4) by $e^{(\mu_1+\dots+\mu_{m-1})t}$ we have for any $x\in R$ and $t>0$

$$|u(x)e^{\mu_m t}|\leqq C\sum_{k=0}^{n}|u(x-mt+kt)|\leqq$$

$$\leqq C\|u\|_{L^\infty(x-mt,\,x+(n-m)t)}\leqq C\|u\|_{L^\infty(x-nt,\,x+nt)}.$$

(Here and in the sequel C denotes an absolute constants not depending on the eigenfunction, not necessarily the same in different places.)

Now put $d(x)\overset{\text{def}}{=}\min(x-a,b-x)$ for $x\in K$. Applying the above estimate we obtain for any $x\in K$ $(t\overset{\text{def}}{=}d(x)/n)$

$$|u(x)e^{\mu_m d(x)/n}|\leqq C\|u\|_{L^\infty(K)},$$

whence

$$|u(x)| \leqq Ce^{-(\operatorname{Re}\mu_m)d(x)/n}\|u\|_{L^\infty(K)} \quad (\forall x\in K). \tag{7}$$

Consider now the case $|\operatorname{Re}\mu_m|>|\operatorname{Re}\mu_{m+1}|$. Then, for any $t<0$

$$\left|\frac{f_k(\mu t)}{e^{(\mu_{m+2}+\ldots+\mu_n)t}}\right| \leqq C \quad \text{if} \quad k\neq m+1$$

and

$$\left|\frac{f_{m+1}(\mu t)}{e^{(\mu_{m+2}+\ldots+\mu_n)t}} - e^{\mu_{m+1}t}\right| \leqq C,$$

Now we obtain for any $x\in R$ and $t<0$

$$|u(x)e^{\mu_{m+1}t}| \leqq C\|u\|_{L^\infty(x+nt,\,x-nt)}$$

and for any $x\in K$ $(t \stackrel{\text{def}}{=} -d(x)/n)$

$$|u(x)e^{-\mu_{m+1}d(x)/n}| \leqq C\|u\|_{L^\infty(K)},$$

whence

$$|u(x)| \leqq Ce^{(\operatorname{Re}\mu_{m+1})d(x)/n}\|u\|_{L^\infty(K)}. \tag{8}$$

Let us now introduce the notation

$$\varrho \stackrel{\text{def}}{=} (1/n)\min\{|\operatorname{Re}\mu_p|:\ 1\leqq p\leqq n\},$$

then in both cases

$$|u(x)| \leqq Ce^{-\varrho d(x)}\|u\|_{L^\infty(K)}. \tag{9}$$

Hence we proceed as in [3]: taking the $L^p(K)$-norm of both sides

$$\|u\|_{L^p(K)} \leqq C(2/p\varrho)^{1/p}\|u\|_{L^\infty(K)},$$

i.e.

$$\|u\|_{L^\infty(K)} \geqq C^p\sqrt[p]{\varrho}\,\|u\|_{L^p(K)}.$$

Hence (6) follows for $\varrho\geqq 1$. On the other hand, the case $\varrho<1$ is trivial and Theorem 2 is proved for $m=0$. The general case follows by induction on m.

References

[1] I. Joó, Upper estimates for the eigenfunctions of the Schrödinger operator, *Acta Sci. Math.*, **44** (1982), 87—93.

[2] I. Joó, V. Komornik, On the equiconvergence of expansions by Riesz bases formed by eigenfunctions of the Schrödinger operator, *Acta Sci. Math.*, **46** (1983), 357—375.

[3] V. Komornik, Lower estimates for the eigenfunctions of the Schrödinger operator, *Acta Sci. Math.*, **44** (1982), 95—98.

[4] V. Komornik, Upper estimates for the eigenfunctions of higher order of a linear differential operator, *Acta Sci. Math.*, **45** (1983), 261—271.

LORÁND EÖTVÖS UNIVERSITY
II. DEPARTMENT OF ANALYSIS
1445 BUDAPEST 8, PF. 323, HUNGARY

Acta Sci. Math., 47 (1948), 205—221

Linear combinations of iterated generalized Bernstein functions with an application to density estimation

WOLFGANG GAWRONSKI and ULRICH STADTMÜLLER

Starting from the classical theorem of Weierstraß (and its various modifications) on approximation of continuous functions by means of Bernstein polynomials (and its generalizations) in this paper a class of discrete and linear operators is developed. These operators are linear combinations of iterates of the original Bernstein type operators being constructed analogously to Fejér—Korovkin operators. Generalizing known results for the classical Bernstein case they approximate smooth functions more closely than the Bernstein type operators. Moreover, related operators for approximating derivatives are developed and these deterministic concepts are applied to probability density estimation for computing the mean square error of certain density estimators.

0. Introduction and summary

It is well known (e.g. [9]; [22]) that classical Bernstein polynomials and its various generalizations and modifications (such as e.g. generalized Bernstein polynomials of Szasz or Baskakov operators) approximate the associated function f with order $O(1/n)$ provided the derivative f' belongs to the class Lip 1. These operators are discrete, linear, and positive. More precisely, they are of form

$$B_n(f;\, x) := \sum_j f\left(\frac{j}{n}\right) p_{jn}(x); \quad f\in C(J), \tag{0.1}$$

where J throughout denotes one of the intervals $\mathbf{R}$, $[0, \infty]$, or $[0, 1]$ for simplicity, and the functions p_{jn} satisfy $p_{jn}(x)\geqq 0$, $x\in J$, $j\in \mathbf{Z}$, $n\in \mathbf{N}$. More generally, in this paper we admit $\{p_{jn}(x)\}_{j=-\infty}^{\infty}$ to be the n-fold convolution of a probability lattice distribution with expectation x (see Section 1) and we refer to (0.1) as a *generalized Bern-*

Received September 13, 1982.

stein function. Then the above order of approximation remains true for (0.1) (e.g. [1]; [2]; [3]; [9]; [12, Ch. 7]; [28]; see also Lemma 3).

In this paper we investigate quantities being related to or derived from the functions (0.1) thereby treating the following topics.

(i) An improvement of the rate of convergence provided f is sufficiently smooth (Section 2),

(ii) approximation of derivatives of f (Section 3), and as an application,

(iii) asymptotic of the mean square error (MSE) of an estimator for a probability density concentrated on J (Section 4).

(i) Dropping the positivity of the operator B_n we mention two methods for increasing the rate of convergence in case of classical Bernstein polynomials. One of them works by forming operators of type

$$(0.2) \qquad L_{r,n} := \sum_{i=1}^{r} a_{ri} B_{d_i n}, \quad 1 \leqq d_1 \leqq \dots \leqq d_r, \quad a_{ri} \in \mathbf{R}$$

(e.g. [4]; [21], where more general singular integral operators with certain differentiability properties are discussed) whereas the second one uses the iterated Bernstein operator of Fejér—Korovkin type

$$(0.3) \qquad D_{r,n} := \sum_{i=1}^{r} \binom{r}{i} (-1)^{i-1} B_n^i$$

[11 and the references given there]. Both approximating functions $L_{r,n}(f; x)$ and $D_{r,n}(f; x)$ are polynomials the approximation order of which is $O(n^{-r})$ provided $f \in C_{2r}[0, 1]$. Besides the increase of the degree $L_{r,n}(f; x)$ has the disadvantage that f has to be evaluated at the points $j/d_i n$, $j=0, \dots, n$, $i=1, \dots, r$, whereas the use of $D_{r,n}(f; x)$ requires the knowledge of f at the distinct nodes $j/n, j=0, \dots, n$ only. We use the second approach due to FELBECKER [11] and extend his result cited above to operators (0.3) based on (0.1) (Theorem 1). In particular this includes the classical Bernstein case treated in [11] which corresponds to $\{p_{jn}(x)\}$ as a binomial distribution and Szasz and Baskakov operators generated by Poisson's and the negative binomial distribution, respectively (see also Section 1).

(ii) If p_{jn}, as a function of $x \in J$, satisfies certain differentiability properties, then e.g. in [13], [30], [31] it was shown that the operators (0.1) are simultaneously approximating, i.e.

$$\left(\frac{d}{dx}\right)^s B_n(f; x) \to f^{(s)}(x),$$

$n \to \infty$, provided f fulfills certain smoothness and growth properties. However for higher derivatives the approximating functions become rather complicated expressions. Hence for approximating the s-th derivative of a function F on J we consider the discrete operators

$$D_n^{(s)}(F;x) := n^s \sum_j p_{jn}(x) \Delta^s F\left(\frac{j}{n}\right), \tag{0.4}$$

where Δ is the forward difference operator acting on j. Then we prove a theorem on uniform approximation and a Voronowskaja property for $D_n^{(s)}$ (Theorems 2, 3).

(iii) If $s=1$, then motivated by (0.4) in [15], [16], [29] a smoothed histogram type estimator was developed for estimating an unknown probability density f concentrated on J. More generally, in Section 3 as an estimator for its r-th derivative we consider

$$\hat{f}_N^{(r)}(x) := n^{r+1} \sum_j p_{jn}(x) \Delta^{r+1} \hat{F}_N\left(\frac{j}{n}\right), \tag{0.5}$$

where $\hat{F}_N$ denotes the empirical distribution function of an iid sample with density f. Extending the results in [15; 16; 29] we compute the exact order of magnitude for the MSE (Theorem 5)

$$E[(\hat{f}_N^{(r)}(x)-f^{(r)}(x))^2]$$

which turns out to be $\sim c \cdot N^{-4/(2r+5)}$ provided the scaling parameter n is chosen subject to $n=n(N)\sim N^{2/5}$, f is smooth enough and satisfies certain growth conditions. (Throughout $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$.) Dropping the property of positivity for an estimator of the density itself we construct an estimator suggested by the deterministic approximation operator (0.3). We replace (0.5) by $(r=0)$

$$\hat{D}_{r,n}(x) := n \sum_j a_{jn}(x) \Delta \hat{F}_N\left(\frac{j}{n}\right) \tag{0.6}$$

as an estimator for $f(x)$, where $a_{jn}(x)$ depends on $p_{jn}(x)$ only. Then the order of the MSE of (0.6) is $N^{-4r/(4r+1)}$ when f is smooth enough again (Theorem 4). Comparable results for the most popular density estimator, the kernel estimator, give the same rate of mean square convergence [23].

In this paper we look at the deterministic approximation theorems from a probabilistic point of view, too. This is expressed by the technical treatment of the proofs where we use e.g. moment inequalities, Tschebyscheff's inequality, local central limit theorems and Edgeworth expansions of lattice distributions.

1. Auxiliary results

In this section we collect and prove some lemmata which are basic for the technical treatment of this paper. We suppose throughout that $\{p_{jn}(x)\}_{j=-\infty}^{\infty}$, $x\in J$, is the n-fold convolution of a lattice probability distribution $\{p_{j1}(x)\}$ concentrated on the integers and satisfying the following conditions:

$$(1.1) \qquad p_{j1} \in C(J),$$

$$(1.2) \qquad \sum_j p_{j1}(x) = 1, \quad \sum_j j p_{j1}(x) = x, \quad \sigma^2 = \sigma^2(x) := \sum_j (j-x)^2 p_{j1}(x),$$

$$(\mathrm{M}_k) \qquad |m|_k(x) := \sum_j |j-x|^k p_{j1}(x) < \infty$$

for some $k \in \mathbf{N}$, $k \geqq 2$, $x \in J$ and the convergence of the series is uniform on compact subsets of J. Further $\{p_{j1}(x)\}$ is assumed to have maximal span equal to 1, and if (M_k) holds, then we denote by

$$m_k(x) := \sum_j (j-x)^k p_{j1}(x)$$

the k-th central moment of $\{p_{j1}(x)\}$. For practical purposes obviously such lattice distributions are of interest for which the p_{j1} are "elementary" functions and the convolutions are easily computable in a closed form. Choices of particular interest are

(i) the binomial distribution

$$p_{jn}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad 0 \leqq x \leqq 1,$$

(ii) Poisson's distribution

$$p_{jn}(x) = \frac{(nx)^j}{j!} e^{-nx}, \quad x \geqq 0,$$

(iii) the negative binomial distribution

$$p_{jn}(x) = \binom{n+j-1}{j} \frac{x^j}{(1+x)^{n+j}}, \quad x \geqq 0,$$

which produce for (0.1) Bernstein polynomials, Szasz and Baskakov operators, respectively. (See also the remarks at the end of Section 4.) Moreover, throughout U is a compact subinterval of J where $\sigma^2(x) \geqq \sigma_0^2 > 0$ holds.

Lemma 1. *Suppose that* (M_k), $k \geqq 2$, *holds, then*

(i) $\sum_j j p_{jn}(x) = nx$,

(ii) $\sum_j (j-nx)^2 p_{jn}(x) = n\sigma^2(x)$,

(iii) $\sum_j |j-nx|^k p_{jn}(x) \leqq A_k |m|_k(x) n^{k/2}$, *where* A_k *is a positive constant de-*

pending only on k,

(iv)*) $\sum_j (j-nx)^k p_{jn}(x) = k! \sum n(n-1)\dots(n-s+1) \prod_{r=1}^{k} \frac{1}{\nu_r!} \left\{\frac{m_r(x)}{r!}\right\}^{\nu_r} =$

$= \sum_{\nu=1}^{[k/2]} a_\nu(x)n^\nu$, *where the non-specified summation is taken over all integer solutions* $(\nu_1, \dots, \nu_k)$ *of the equations* $\nu_1+2\nu_2+\dots+k\nu_k=k$, $s=\nu_1+\dots+\nu_k$; *moreover, if* $k=2r$ *is even, then*

$$a_r(x) = \frac{(2r)!}{r!\,2^r}\sigma^{2r}(x).$$

Proof. (i), (ii) are trivial and (iii) is a form of Marcinkiewicz's inequality (e.g. [25, p. 41]); explicit bounds are given in [24, p. 60], [8]. The first equality in (iv) is obtained by using the k-th derivative of the characteristic function of $\{p_{jn}(x)\}$ computed via [24, Lemma 2, p. 135]. From the latter form the representation as polynomial in n is immediate. This polynomial has degree at most $[k/2]$, since $m_1(x)=0$, by (1.2) and because $\nu_1+2\nu_2+\dots+k\nu_k=k$, $\nu_1+\dots+\nu_k>k/2$ imply that $\nu_1\geqq 1$. Finally the form of $a_r(x)$ in case $k=2r$ is given in [7, Corollary 3 of Theorem 2, p. 294].

Using notations and properties of the difference operator in [12, p. 221] and a local central limit theorem [24, Theorem 17, p. 207, see also pp. 9, 139] in [14, Lemma 1] the following lemma is proved.

Lemma 2. (i) *Suppose that* (M_3) *holds,* $m\in\mathbf{N}$, $\delta_n>0$ *and* $\delta_n\sqrt{n}\to\infty$. *Then for* $x\in U$ *we have*

$$\sum_{|j/n-x|\leqq\delta_n} p_{jn}(x)^m = \frac{1}{(2\pi\sigma^2(x)n)^{(m-1)/2}\sqrt{m}}(1+o(1)), \quad n\to\infty,$$

the o-term being independent of $x\in U$.

(ii) *If* (M_k) *holds with* r, $m\in\mathbf{N}$, $k\geqq r+2$ *and* $\delta_n>0$ *with* $\delta_n\sqrt{n}\to\infty$ *as* $n\to\infty$, *then for* $x\in U$ *we have*

$$\sum_{|j/n-x|\leqq\delta_n} |\Delta^r p_{j-r,n}(x)|^m = \frac{\sigma\sqrt{n}\,c_{r,m}}{(\sqrt{2\pi}\,\sigma^{r+1}n^{(r+1)/2})^m}(1+o(1)), \quad n\to\infty.$$

Again the o-term is independent of $x\in U$ *and*

$$c_{r,m} := \int_{-\infty}^{\infty} |H_r(y)|^m e^{-my^2/2}\,dy,$$

where H_r *is the r-th Hermite polynomial defined e.g. in* [24, p. 139].

*) For $\xi\in\mathbf{R}$, $[\xi]$ denotes the largest integer not exceeding ξ, as customary.

Finally we mention an approximation theorem together with a Voronowskaja property for the operators B_n in (0.1) which is well known and has been treated in the literature in various modified versions (e.g. [1], [2], [3], [5], [12], [19], [20], [21], [28], [30], [31]) since it can be proved along standard lines we omit a proof.

Lemma 3. (i) *If* (M_k) *holds for some* $k \geqq 2$, *and if* $f \in C(J)$ *satisfies* $f(x) = O(x^k)$, $|x| \to \infty$, *then*

$$\lim_{n\to\infty} B_n(f; x) = f(x) \tag{1.3}$$

for all $x \in J$. *Moreover, the convergence is uniform on compact subsets of* J.

(ii) *If* (M_k) *holds for some* $k \geqq 3$, *and if* $f \in C_2(J)$ *satisfies* $f(x) = O(x^k)$, $|x| \to \infty$, *then*

$$\lim_{n\to\infty} n\{B_n(f; x) - f(x)\} = \frac{\sigma^2(x)}{2} f''(x) \tag{1.4}$$

for all $x \in J$. *Again the convergence is uniform on compact subsets of* J.

Considering instead of (0.1) the modification

$$B_n^*(f; x) := n \sum_j p_{jn}(x) \int_{j/n}^{(j+1)/n} f(y)\,dy \tag{1.5}$$

which can be looked at as generalized Kantorovich functions (cf. [5], [6], [20]) Lemma 3 remains true provided the right hand side of (1.4) is replaced by

$$B^*(f; x) := \frac{1}{2}\left(f'(x) + \sigma^2(x) f''(x)\right). \tag{1.6}$$

In the sequel for $f \in C_2(J)$ we use the notation

$$B(f; x) := \frac{\sigma^2(x)}{2} f''(x). \tag{1.7}$$

2. Iterated generalized Bernstein functions

In this section we treat topic (i) mentioned in the introduction. That is, generalizing Lemma 3 and extending partially the results in [11] we improve the rate of convergence in (1.3). Under (M_ν), $\nu \geqq 2$, we consider the growth condition

$$m_\nu(x) = O(x^\nu), \quad |x| \to \infty, \tag{2.1}$$

which in particular is satisfied for the examples cited in Section 1 and more generally, when $\sigma^2(x)$ is a polynomial of degree at most 2 and p_{j1} satisfies the differential equation $\sigma^2(x) p'_{j1}(x) = p_{j1}(x)(j - x)$ (cf. [13], [21], [30]). Then for functions f defined on J

and satisfying $f(x)=O(x^\nu)$, $|x|\to\infty$, in case of an unbounded interval J the *iterated generalized Bernstein functions*

$$B_n^r(f),\quad r\in\mathbf{N}_0, \tag{2.2}$$

are well defined and continuous on J. Here $B_n^0:=I$ is the identity operator, and $B_n^{r+1}(f):=B_n(B_n^r(f))$, $r\in\mathbf{N}_0$. Further we have by (2.1) and Lemma 1 (iii)

$$B_n^r(f;x) = O(x^\nu),\quad |x|\to\infty,\quad r\in\mathbf{N}_0, \tag{2.3}$$

the O-constant being independent of n. Moreover, $D_{k,n}$ (defined in (0.3)) can be written as

$$D_{k,n} = I-(I-B_n)^k,\quad k\in\mathbf{N}. \tag{2.4}$$

Then we prove

Theorem 1. *Suppose that* (M_k) *holds for some* $k\geqq 2r+1$, $r\in\mathbf{N}$, *and* m_ν *satisfies* (2.1) *and* $m_\nu\in C_{2r-2}(J)$ *for* $\nu=2,3,\dots,2r$. *Moreover, assume that* $f\in C_{2r}(J)$ *and* $f^{(2r)}(x)=O(x^{k-2r})$, $|x|\to\infty$. *Then*

$$\lim_{n\to\infty} n^r\big(D_{r,n}(f;x)-f(x)\big) = (-1)^{r-1}B^r(f;x)$$

for all $x\in J$. *Further the convergence is uniform on compact subintervals of* J. *(The powers of* B *in* (1.7) *are inductively defined in the same way as those of* B_n.*)*

Proof. We proceed by induction with respect to r (see [11]). If $r=1$, then Theorem 1 is contained in Lemma 3 (ii). Hence we assume the assertion to be true for $r-1$, $r\geqq 2$. Let $K=[a,b]\subseteqq J$ and for $\delta>0$ we use the notation $K_\delta:=[a-\delta, b+\delta]$. Then we have

$$f(y) = \sum_{\nu=0}^{2r}(y-x)^\nu\frac{f^{(\nu)}(x)}{\nu!}+(y-x)^{2r}\varrho(y-x),\quad x\in K \tag{2.5}$$

where $\varrho(h)\to 0$, if $h\to 0$ and

$$(y-x)^{2r}\varrho(y-x) = O\big((y-x)^k\big),\quad |y|\to\infty \tag{2.6}$$

uniformly in $x\in K$. Now the Taylor series expansion (2.5) gives (use (1.2))

$$(B_n-I)(f;x) = \sum_{\nu=2}^{2r}\frac{f^{(\nu)}(x)}{\nu!}\sum_j\left(\frac{j}{n}-x\right)^\nu p_{jn}(x)+\sum_j\left(\frac{j}{n}-x\right)^{2r}\varrho\left(\frac{j}{n}-x\right)p_{jn}(x) =$$

$$=: \sum_{\nu=2}^{2r}\frac{1}{\nu!}\frac{f^{(\nu)}(x)}{n^\nu}\sum_j (j-nx)^\nu p_{jn}(x)+\xi_n(x).$$

Using (2.1), the differentiability properties of m_ν and the growth restriction on $f^{(2r)}$, by Lemma 1 (iv) we can write the latter identity as

$$(B_n-I)(f;x) = \sum_{s=1}^{2r-1}\frac{1}{n^s}g_s(x)+\xi_n(x),$$

where the functions g_s are independent of n and satisfy the conditions

$$g_s(x) = O(x^k), \quad |x| \to \infty, \tag{2.7}$$

and

$$g_s \in \begin{cases} C_{2(r-s)}(J), & 1 \leqq s \leqq r \\ C(J), & r < s \leqq 2r-1 \end{cases} \tag{2.8}$$

with $g_1(x) = B(f; x)$. Hence it follows that

$$\begin{aligned} n^r(B_n - I)^r(f; x) &= n^{r-1}(B_n - I)^{r-1}B(f; x) + \\ &+ \sum_{s=2}^{2r-1} n^{r-s}(B_n - I)^{r-1}(g_s; x) + (B_n - I)^{r-1}(n^r \xi_n; x). \end{aligned} \tag{2.9}$$

(Note that B_n can be applied to g_s, ξ_n, and $B(f)$, by (2.7), (2.6) and (2.3).)

Next, for functions $f_n \in C(J)$ satisfying

$$f_n(x) = O(x^k), \quad |x| \to \infty \tag{2.10}$$

uniformly in n we get (see (2.3); $M > 0$)

$$\begin{aligned} \sup_{x\in K} |(B_n - I)(f_n; x)| &\leqq \sup_{x\in K} |f_n(x)| + \sup_{x\in K} |B_n(f_n; x)| \leqq \\ \leqq \sup_{x\in K} |f_n(x)| + \sup_{x\in K} \left| \sum_{j/n\in K_\delta} f_n\left(\frac{j}{n}\right) p_{jn}(x) \right| &+ \sup_{x\in K} \left| \sum_{j/n\notin K_\delta} f_n\left(\frac{j}{n}\right) p_{jn}(x) \right| \leqq \\ \leqq 2 \sup_{x\in K_\delta\cap J} |f_n(x)| &+ M \sup_{x\in K} \sum_{j/n\notin K_\delta} \left(\left|\frac{j}{n}\right|^k + 1 \right) p_{jn}(x) \leqq \\ \leqq 2 \sup_{x\in K_\delta\cap J} |f_n(x)| + M \sup_{x\in K} &\left(\left\{ \sum_{v=0}^{k} \binom{k}{v} \frac{|x|^{k-v}}{\delta^{k-v}} + \frac{1}{\delta^k} \right\} \sum_j \left| \frac{j}{n} - x \right|^k p_{jn}(x) \right) = \\ &= 2 \sup_{x\in K_\delta\cap J} |f_n(x)| + O\left(\frac{1}{n^{k/2}}\right), \end{aligned}$$

by Lemma 1 (iii). Further, by (2.3) and (2.10) we may apply this estimate to $f_n(x) = (B_n - I)^{r-2}(f; x)$ and thus we obtain inductively (observe (2.7), (2.8))

$$\begin{aligned} &\sup_{x\in K} |(B_n - I)^{r-1}(g_s; x)| \leqq \\ &\leqq \begin{cases} 2^{s-2} \sup\limits_{x\in K_{(s-2)\delta}\cap J} |(B_n - I)^{r-s+1}(g_s; x)| + O\left(\frac{1}{n^{k/2}}\right), & 2 \leqq s \leqq r \\ 2^{r-1} \sup\limits_{x\in K_{(r-1)\delta}\cap J} |g_s(x)| + O\left(\frac{1}{n^{k/2}}\right), & r < s \leqq 2r-1 \end{cases} \end{aligned}$$

as $n \to \infty$. Now from (2.9), by the induction hypothesis, we have

$$n^{r-1}(B_n - I)^{r-1}B(f; x) \to B^{r-1}(B(f; x)) = B^r(f; x)$$

and, since $k \geqq 2r+1$,

$$n^{r-s}(B_n-I)^{r-1}(g_s; x) \to 0, \quad s \geqq 2,$$

as $n \to \infty$, uniformly on K. Finally we conclude from (2.5), (2.6), and Lemma 1 (iii) that $(M, \varepsilon, \delta > 0)$

$$n^r |\xi_n(x)| \leqq \varepsilon + M n^{r-k} \sum_{|j-nx|>\delta n} |j-nx|^k p_{jn}(x) \leqq \varepsilon + \frac{M}{\sqrt{n}} \leqq 2\varepsilon$$

when n is large enough. This and (2.4) complete the proof.

In later applications (see Section 4) we need a modification of Theorem 1 for the generalized Kantorovich operators B_n^* (defined in (1.5)). Putting

$$D_{k,n}^* := \sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} B_n^{*i}, \quad k \in \mathbf{N}, \tag{0.3)*$$

we have

$$D_{k,n}^* = I-(I-B_n^*)^k, \quad k \in \mathbf{N}. \tag{2.4)*$$

Since the following theorem is proved along the same lines as the preceeding one we omit its proof and only state

Theorem 1*. *Under the assumptions of Theorem* 1 *we have*

$$\lim_{n\to\infty} n^r (D_{r,n}^*(f; x) - f(x)) = (-1)^{r-1} B^{*r}(f; x)$$

for all $x \in J$. *Again the convergence is uniform on compact subsets of* J.

3. Approximation of derivatives

In this section we treat topic (ii) mentioned in the introduction; that is, we prove an approximation theorem for the operators (0.4) together with a Voronowskaja property. In the sequel Δ denotes the difference operator defined by $\Delta a_j := a_{j+1} - a_j$ acting on a sequence $\{a_j\}$ (e.g. [12, p. 221]). For differences of higher order we have

$$\Delta^r a_j = \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^{r-\nu} a_{j+\nu}, \quad r \in \mathbf{N}_0, \tag{3.1}$$

where $\Delta^0 a_j := a_j$.

Theorem 2. *Suppose that* (M_k) *holds for some* $k \geqq 2$ *and* $F \in C_s(J)$, $s \geqq 0$, *satisfies* $F^{(s)}(x) = O(x^k)$, $|x| \to \infty$. *Then*

$$\lim_{n\to\infty} D_n^{(s)}(F; x) = F^{(s)}(x)$$

for all $x \in J$ *and the convergence is uniform on compact subsets of* J.

Proof. First we note that for $F \in C_s(\mathbf{R})$

$$\Delta^s F\left(\frac{j}{n}\right) = \frac{1}{n^s} F^{(s)}(\xi_{jn}) \tag{3.2}$$

with some $\xi_{jn} \in [j/n, (j+s)/n]$. Extending F suitably from J on $\mathbf{R}$ we have, by (3.2) $(\varepsilon, \delta > 0)$

$$|D_n^{(s)}(F; x) - F^{(s)}(x)| = \left|\sum_j \left(F^{(s)}(\xi_{jn}) - F^{(s)}(x)\right) p_{jn}(x)\right| \leqq$$

$$\leqq \varepsilon + \sum_{|\xi_{jn} - x| > \delta} |F^{(s)}(\xi_{jn}) - F^{(s)}(x)| p_{jn}(x).$$

Further, restricting x to a compact subset of J with positive constants M, M' we get (see Lemma 1 (iii))

$$|D_n^{(s)}(F; x) - F^{(s)}(x)| \leqq \varepsilon + M \sum_{|\xi_{jn} - x| > \delta} |\xi_{jn} - x|^k p_{jn}(x) \leqq$$

$$\leqq \varepsilon + M \sum_j \left(\frac{s}{n} + \left|\frac{j}{n} - x\right|\right)^k p_{jn}(x) \leqq \varepsilon + \frac{M'}{n^{k/2}} \leqq 2\varepsilon$$

if n is large enough. This completes the proof.

The exact rate of convergence is given by the following Voronowskaja property.

Theorem 3. *Suppose that* (M_k) *holds for some* $k \geqq 3$ *and* $F \in C_{s+2}(J)$, $s \geqq 0$, *satisfies* $F^{(s+2)}(x) = O(x^{k-2})$, $|x| \to \infty$. *Then*

$$\lim_{n \to \infty} n\left(D_n^{(s)}(F; x) - F^{(s)}(x)\right) = \frac{1}{2}\left(sF^{(s+1)}(x) + \sigma^2(x) F^{(s+2)}(x)\right) \tag{3.3}$$

for all $x \in J$, *the convergence being uniform on compact subsets of* J.

Remark. If $s = 0, 1$, then the right hand side of (3.3) can be written as $B(F; x)$ and $B^*(F'; x)$, respectively. This exhibits Theorem 3 as a generalization of Lemma 3 (ii) and the corresponding analogue for B_n^*. (See the remarks following Lemma 3.)

Proof of Theorem 3. Extending F suitably from J on $\mathbf{R}$ if necessary, since $F \in C_{s+2}(J)$, we have $(0 \leqq \nu \leqq s)$

$$F\left(\frac{j+\nu}{n}\right) = \sum_{\mu=0}^{s+1} \frac{1}{\mu!} F^{(\mu)}\left(\frac{j}{n}\right)\left(\frac{\nu}{n}\right)^\mu + \frac{1}{(s+2)!} F^{(s+2)}(\xi_{j\nu})\left(\frac{\nu}{n}\right)^{s+2}$$

with $j/n \leqq \xi_{j\nu} \leqq (j+\nu)/n$ and further, by (3.1),

$$\Delta^s F\left(\frac{j}{n}\right) = \sum_{\mu=0}^{s+1} \frac{F^{(\mu)}(j/n)}{n^\mu \mu!} \sum_{\nu=0}^{s} \binom{s}{\nu} (-1)^{s-\nu} \nu^\mu +$$

$$+ \frac{1}{n^{s+2}(s+2)!} \sum_{\nu=0}^{s} \binom{s}{\nu} (-1)^{s-\nu} \nu^{s+2} F^{(s+2)}(\xi_{j\nu}).$$

Since, by (3.1), foı $a_j=j^\mu$ and $j=0$

$$\frac{1}{s!}\sum_{v=0}^{s}\binom{s}{v}(-1)^{s-v}v^{\mu}=\frac{1}{s!}\Delta^s j^\mu=\begin{cases}0, & 0\leqq\mu<s,\\ 1, & \mu=s,\\ \binom{s+1}{2}, & \mu=s+1,\end{cases}$$

we obtain

$$D_n^{(s)}(F;x)=\sum_j F^{(s)}\left(\frac{j}{n}\right)p_{jn}(x)+\frac{s}{2n}\sum_j F^{(s+1)}\left(\frac{j}{n}\right)p_{jn}(x)+$$

$$+\frac{1}{n^2(s+2)!}\sum_{v=0}^{s}\binom{s}{v}(-1)^{s-v}v^{s+2}\sum_j F^{(s+2)}(\xi_{jv})p_{jn}(x)=$$

$$=B_n(F^{(s)};x)+\frac{s}{2n}B_n(F^{(s+1)};x)+O\left(\frac{1}{n^2}\right)$$

the O-term being independent of x as long as x is restricted to a compact subset of J. Now Lemma 3 completes the proof.

4. MSE for density estimators of Bernstein type

In this section let f be an unknown probability density concentrated on a known interval J. Starting from the empirical distribution function $\hat{F}_N$ based on iid observations $X_1, \ldots, X_N$ having density f, Theorem 2 shows that, $\hat{f}_N^{(r)}(x)$, defined in (0.5), is an asymptotically unbiased estimator for the r-th derivative $f^{(r)}(x)$ provided f satisfies certain growth and smoothness conditions. If $r=0$, then for various particular cases in [15], [16], [29] the asymptotic of the MSE was computed. Asymptotic distributions for $\hat{f}_N^{(0)}$ have been derived in [27]. Based on Theorem 1* and Theorem 3 now we accelerate the mean square convergence, when $r=0$, and determine the asymptotic behaviour of the MSE for $\hat{f}_N^{(r)}(x)$ if $r\geqq 0$.

First dropping the positivity of $\hat{f}_N^{(0)}$ and motivated by Theorem 1* we consider

$$\hat{D}_{r,n}(x):=n\sum_j a_{jn}(x)\Delta\hat{F}_N\left(\frac{j}{n}\right) \tag{4.1}$$

with

$$a_{jn}(x):=\sum_{i=1}^{r}\binom{r}{i}(-1)^{i-1}B_n^{*i-1}(p_{jn};x) \tag{4.2}$$

as an estimator for $f(x)$.

Theorem 4. *Suppose that* (M_k) *holds for some* $k\geqq 2r+1$, $r\in\mathbf{N}$, m_v *satisfies* (2.1) *and* $m_v\in C_{2r-2}(J)$ *for* $v=2,3,\ldots,2r$. *Moreover assume that* $f\in C_{2r}(J)$ *and* $f^{(2r)}(x)=O(x^{k-2r})$, $|x|\to\infty$.

(i) *If* $\sigma^2(x)>0$, *then*

$$E((\hat{D}_{r,n}(x)-f(x))^2) = \left\{\frac{B^{*r}(f;x)}{n^r}\right\}^2 + o\left(\frac{1}{n^{2r}}\right) + \frac{1}{N}V_n(x) \quad as \quad n\to\infty,$$

where

$$V_n(x) = \frac{f(x)\sqrt{n}}{2\sqrt{\pi}\,\sigma(x)} + o(\sqrt{n}), \quad if \quad r=1$$

$$|V_n(x)| \leqq \frac{(2^r-1)^2}{\sqrt{2\pi}\,\sigma_0} f(x)\sqrt{n} + O(1), \quad if \quad r>1.$$

Here the remainder terms hold uniformly on $U\subseteqq J$ *and* $\sigma(x)\geqq\sigma_0$ *for* $x\in U$.

(ii) *If* $\sigma^2(x)=0$, *then as* $n\to\infty$

$$E((\hat{D}_{r,n}(x)-f(x))^2) = \left\{\frac{f^{(r)}(x)}{2^r n^r}\right\}^2 + o\left(\frac{1}{n^{2r}}\right) + \frac{f(x)n}{N} + \frac{1}{N}o(n).$$

Proof. We decompose

$$E((\hat{D}_{r,n}(x)-f(x))^2) = (E(\hat{D}_{r,n}(x))-f(x))^2 + \operatorname{Var}(\hat{D}_{r,n}(x))$$

as a sum of bias squared and a variance term. If F denotes the distribution function of f, then an application of Theorem 1* yields

$$E(\hat{D}_{r,n}(x)) = n\sum_j a_{jn}(x)\Delta F\left(\frac{j}{n}\right) = \sum_{i=1}^{r}\binom{r}{i}(-1)^{i-1}B_n^{*i}(f;x) = \tag{4.3}$$

$$= D^*_{r,n}(f;x) = f(x) + \frac{(-1)^{r-1}}{n^r}B^{*r}(f;x) + o\left(\frac{1}{n^r}\right),$$

where the o-term holds uniformly on compact subsets of J. For the variance we note that

$$\operatorname{Cov}\left(\Delta\hat{F}_N\left(\frac{j}{n}\right), \Delta\hat{F}_N\left(\frac{k}{n}\right)\right) = \frac{1}{N}\Delta F\left(\frac{j}{n}\right)\left(\delta_{jk}-\Delta F\left(\frac{k}{n}\right)\right) \tag{4.4}$$

and obtain

$$\operatorname{Var}(\hat{D}_{r,n}(x)) = n^2\sum_{j,k}\operatorname{Cov}\left(\Delta\hat{F}_N\left(\frac{j}{n}\right), \Delta\hat{F}_N\left(\frac{k}{n}\right)\right)a_{jn}(x)a_{kn}(x) = \tag{4.5}$$

$$= \frac{n^2}{N}\left\{\sum_j a_{jn}(x)^2\Delta F\left(\frac{j}{n}\right) - \left(\sum_j a_{jn}(x)\Delta F\left(\frac{j}{n}\right)\right)^2\right\}.$$

(i) Suppose that $\sigma^2(x)>0$. If $r=1$, then $a_{jn}(x)=p_{jn}(x)$ and, by Lemma 2(i) it is easily shown that

$$n^{3/2}\sum_j p_{jn}(x)^2\Delta F\left(\frac{j}{n}\right) = \frac{f(x)}{2\sqrt{\pi}\,\sigma(x)} + o(1), \quad n\to\infty, \tag{4.6}$$

uniformly on U. If $r \geqq 2$, then we use a local central limit theorem (see formula (1.4) in [14]; or Theorem 1 in [24, p. 207]) and obtain

$$B_n^{*i}(p_{jn};\, x) \leqq \frac{1}{\sqrt{2\pi n}\,\sigma_0} + O\left(\frac{1}{n}\right), \quad i \in \mathbf{N}_0,$$

where the O-term holds uniformly in $x \in U$. From this we get, by (4.2),

$$|a_{jn}(x)| \leqq \frac{2^r - 1}{\sqrt{2\pi n}\,\sigma_0} + O\left(\frac{1}{n}\right)$$

and thus, by the remarks following Lemma 3,

$$\text{(4.7)} \qquad n^{3/2} \sum_j a_{jn}(x)^2 \Delta F\left(\frac{j}{n}\right) \leqq \left\{\frac{(2^r-1)}{\sqrt{2\pi}\,\sigma_0} + O\left(\frac{1}{\sqrt{n}}\right)\right\} \sum_{i=1}^{r} \binom{r}{i} B_n^{*i}(f;\, x) =$$
$$= \frac{(2^r-1)}{\sqrt{2\pi}\,\sigma_0} f(x) + O\left(\frac{1}{\sqrt{n}}\right)$$

uniformly on U. Now a combination of (4.3), (4.5), (4.6), (4.7) completes the proof of part (i).

(ii) In case $\sigma^2(x)=0$, we have $p_{jn}(x)=\delta_{j,nx}$ for some $j \in \mathbf{Z}$, δ_{jk} being Kronecker's symbol. This implies that $a_{jn}(x)=p_{jn}(x)=\delta_{j,nx}$, by (4.2). Using (1.6), (4.3) and (4.4) we find part (ii).

In case $\sigma^2(x)>0$ obviously the choice $n=n(N) \sim cN^{2/(4r+1)}$, $c>0$, $N \to \infty$, yields the estimate $(r>1)$

$$E(\hat{D}_{r,n}(x)-f(x))^2 = O(N^{-4r/(4r+1)}), \quad N \to \infty.$$

For corresponding kernel estimators (cf. [18, section 4]) $N^{-4r/(4r+1)}$ is the exact order of magnitude. By more careful estimates of $V_n(x)$ the constant involved in the leading term could be reduced.

Finally we extend Theorem 1 in [16] by

Theorem 5. *Suppose that* (M_k) *holds for some* $k \geqq \max(3, r+2)$, $r \in \mathbf{N}_0$. *Further, assume that* $f \in C_{r+2}(J)$ *and* $f^{(r+2)}(x)=O(x^{k-2})$, $|x| \to \infty$.

i) *If* $\sigma^2(x) > 0$, *then*

$$E((\hat{f}_N^{(r)}(x)-f^{(r)}(x))^2) = \left\{\frac{(r+1)f^{(r+1)}(x)+\sigma^2(x)f^{(r+2)}(x)}{2n}\right\}^2 + o\left(\frac{1}{n^2}\right) +$$
$$+ \frac{f(x)c_{r,2}n^{r+1/2}}{2\pi\sigma^{2r+1}(x)N} + \frac{1}{N}o(n^{r+1/2}),$$

as $n \to \infty$, *where* $c_{r,2}$ *is defined in Lemma 2. Again the* o*-terms hold uniformly on* U.

(ii) *If* $\sigma^2(x)=0$, *then*

$$E((\hat{f}_N^{(r)}(x)-f^{(r)}(x))^2) = \left(\frac{f^{(r)}(x)}{2n}\right)^2 + o\left(\frac{1}{n^2}\right) + \frac{f(x)4^r n^{2r+1}}{N} + \frac{1}{N}O(n^{2r}), \quad n\to\infty.$$

Proof. Due to the standard decomposition

$$E((\hat{f}_N^{(r)}(x)-f^{(r)}(x))^2) = (E(\hat{f}_N^{(r)}(x))-f^{(r)}(x))^2 + \operatorname{Var}(\hat{f}_N^{(r)}(x))$$

we treat each summand separately. By Theorem 3 $(s=r+1)$ and (0.5) we get $(F'(x)=f(x))$

(4.8) $$E(\hat{f}_N^{(r)}(x)) = D_n^{(r+1)}(F;x) = f^{(r)}(x)+$$
$$+\frac{1}{2n}((r+1)f^{(r+1)}(x)+\sigma^2(x)f^{(r+2)}(x))+o\left(\frac{1}{n}\right)$$

uniformly on compact subsets of J. For evaluating the variance we use partial summation (see also [14]) and obtain from (0.5)

$$\hat{f}_N^{(r)}(x) = (-1)^r n^{r+1}\sum_j \Delta^r p_{j-r,n}(x)\Delta\hat{F}_N\left(\frac{j}{n}\right).$$

Hence, by (4.4), we have

$$\operatorname{Var}(\hat{f}_N^{(r)}(x)) = \frac{n^{2(r+1)}}{N}\sum_j (\Delta^r p_{j-r,n}(x))^2\Delta F\left(\frac{j}{n}\right) - \frac{1}{N}(E(\hat{f}_N^{(r)}(x)))^2 =: \mathrm{I}-\mathrm{II},$$

say. For II we have by (4.8)

(4.9) $$\mathrm{II} = \frac{1}{N}O(1), \quad n\to\infty.$$

Next, by the continuity of f at $x\in U$ $(\varepsilon, \delta>0)$ we get

$$\mathrm{I}-\frac{n^{2r+1}f(x)}{N}\sum_{|j/n-x|\le\delta}(\Delta^r p_{j-r,n}(x))^2 =$$
$$=\frac{n^{2r+1}}{N}\sum_{|j/n-x|\le\delta} n\int_{j/n}^{(j+1)/n}(f(y)-f(x))\,dy\,(\Delta^r p_{j-r,n}(x))^2+$$
$$+\frac{n^{2(r+1)}}{N}\sum_{|j/n-x|>\delta}(\Delta^r p_{j-r,n}(x))^2\Delta F\left(\frac{j}{n}\right) =: \mathrm{I}'+\mathrm{II}',$$

say, and

(4.10) $$|\mathrm{I}'| \le \varepsilon\frac{n^{2r+1}}{N}\sum_{|j/n-x|\le\delta}(\Delta^r p_{j-r,n}(x))^2.$$

Writing $(\Delta F(j/n)\le 1)$

$$\mathrm{II}' \le \frac{n^{2(r+1)}}{N}\sum_{\nu,\mu=1}^{r}\binom{r}{\nu}\binom{r}{\mu}\sum_{|j/n-x|>\delta}p_{j+\nu-r,n}(x)p_{j+\mu-r,n}(x),$$

the use of Cauchy's inequality combined with Lemma 1 (iii) yields

$$\mathrm{II}' = \frac{1}{N} O(n^r), \quad n \to \infty.$$

Putting this together with (4.8)—(4.10), and Lemma 2 (ii) we have established part (i).

In case $\sigma^2(x)=0$, direct computation of I (note that $p_{jn}(x)=\delta_{j,nx}$ and x is an integer) yields

$$\mathrm{I} = \frac{n^{2r+2}}{N} \sum_j \Delta F\left(\frac{j}{n}\right) \left\{ \sum_{v=0}^{r} \binom{r}{v} (-1)^{r-v} \delta_{j+v-r,nx} \right\}^2 =$$

$$= \frac{n^{2r+2}}{N} \sum_j \Delta F\left(\frac{j}{n}\right) \binom{r}{r+nx-j}^2 = \frac{n^{2r+2}}{N} \sum_{v=0}^{r} \Delta F\left(\frac{v}{n}+x\right) \binom{r}{r-v}^2 =$$

$$= \frac{n^{2r+1}}{N} f(x) 4^r + \frac{1}{N} O(n^{2r}), \quad n \to \infty,$$

thereby finishing the proof of part (ii).

Looking at the case $\sigma^2(x)>0$ we see that the "optimal" choice $n=n(N)\sim \sim cN^{2/(2r+5)}$, $c>0$, $N\to\infty$ leads to the exact order of magnitude $N^{-4/(2r+5)}$ for the MSE of $\hat{f}_N^{(r)}$. Comparable results for the classical kernel estimator give the same rate of mean square convergence (e.g. [18]). Finally it should be pointed out that in particular the estimators (0.5) for the derivatives derived from (0.4) seem suitable rather than estimators obtained from $\hat{f}_N^{(0)}$ by differentiation with respect to x; for such estimators have complicated forms, if r is large. However in practice the computation of the coefficients of $p_{jn}(x)$ in (0.5) essentially requires only the evaluation of differences for a sequence of integers.

In this paper we have considered approximating operators and density estimators constructed by a lattice distribution. Motivated by a local central limit theorem another example is suggested by

$$p_{jn}(x) := \frac{1}{\sqrt{2\pi n}} e^{-(j-nx)^2/2n}; \quad j\in\mathbf{Z},\ x\in\mathbf{R},$$

which can be shown (see [16], [17]) to be a "good" approximation of a lattice distribution with mean nx and variance n (i.e. $\sigma^2(x)\equiv 1$). This approach leads to Favard operators for (0.1) [10], [17] for which the topics of this paper can be discussed in a similar way.

References

[1] M. Arató and A. Rényi, Probabilistic proof of a theorem of the approximation of continuous functions by means of generalized Bernstein polynomials, *Acta Math. Acad. Sci. Hungar.*, **8** (1957), 91—98.

[2] V. A. Baskakov, An example of a sequence of linear positive operators in the space of continuous functions, *Doklady Akad. Nauk. SSSR*, **113** (1957), 249—251.

[3] S. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités, *Commun. Soc. Math. Kharkow*, (2) **13** (1912—13), 1—2.

[4] P. Butzer, Linear combinations of Bernstein polynomials, *Canad. J. Math.*, **5** (1953), 559—567.

[5] P. Butzer, On the extensions of Bernstein polynomials to the infinite interval, *Proc. Amer. Math. Soc.*, **5** (1954), 547—553.

[6] P. Butzer, Summability of generalized Bernstein polynomials. I, *Duke Math. J.*, **22** (1955), 617—622.

[7] Y. S. Chow and H. Teicher, *Probability theory, independence, interchangeability, martingales,* Springer-Verlag (New York—Heidelberg—Berlin, 1978).

[8] S. Dharmadhikari and K. Jogdeo, Bounds on moments of certain random variables, *Ann. Math. Statist.*, **40** (1969), 1506—1508.

[9] R. De Vore, *The approximation of continuous functions by positive linear operators,* Lecture notes in mathematics, No. 293, Springer-Verlag (Berlin—Heidelberg—New York, 1972).

[10] J. Favard, Sur les multiplicateurs d'interpolation, *J. Math. Pures Appl.*, **23** (1944), 219—247.

[11] F. Felbecker, Linearkombinationen von iterierten Bernsteinoperatoren, *Manuscripta Math.*, **29** (1979), 229—248.

[12] W. Feller, *An introduction to probability theory and its applications,* Vol. II, J. Wiley & Sons (New York—London—Sidney—Toronto, 1971).

[13] W. Gawronski, *Verallgemeinerte Bernsteinfunktionen und Schätzung einer Wahrscheinlichkeitsdichte,* Habilitationsschrift, Ulm, 1980.

[14] W. Gawronski, Strong laws for density estimators of Bernstein type, to appear.

[15] W. Gawronski and U. Stadtmüller, On density estimation by means of Poisson's distribution. *Scand. J. Statist.*, **7** (1980), 90—94.

[16] W. Gawronski and U. Stadtmüller, Smoothing histograms by means of lattice and continuous distributions, *Metrika,* **28** (1981), 155—164.

[17] W. Gawronski and U. Stadtmüller, Approximation of continuous functions by generalized Favard operators, *J. Approx. Theory,* **34** (1982), 384—396.

[18] T. Gasser and H.-G. Müller, Optimal convergence properties of kernel estimates of derivatives of a density function, in: *Smoothing techniques for curve estimation,* Lecture notes in mathematics, No. 757, Springer-Verlag (Berlin—Heidelberg—New York, 1979).

[19] S. Karlin and Z. Ziegler, Iteration of positive approximation operators, *J. Approx. Theory,* **3** (1970), 310—339.

[20] G. G. Lorentz, *Bernstein polynomials,* University of Toronto Press (Toronto, 1953).

[21] C. May, Saturation and inverse theorems for combinations of a class of exponential-type operators, *Canad. J. Math.*, **28** (1976), 1224—1250.

[22] M. W. Müller, *Approximationstheorie,* Akad. Verlagsgesellschaft (Wiesbaden, 1978).

[23] E. Parzen, On estimation of a probability density and mode, *Ann. Math. Statist.*, **33** (1962), 1065—1076.

[24] V. V. Petrov, *Sums of independent random variables*, Springer-Verlag (Berlin—Heidelberg—New York, 1975).

[25] P. Révész, *Gesetze der großen Zahlen*, Birkhäuser-Verlag (Basel—Stuttgart, 1968).

[26] R. S. Singh, Improvement on some known nonparametric uniformly consistent estimators of derivatives of a density, *Ann. Statist.*, **5** (1977), 394—399.

[27] U. Stadtmüller, Asymptotic distributions of smoothed histograms, *Metrika*, **30** (1983), 145—158.

[28] O. Szasz, Generalization of S. Bernstein's polynomials to the infinite interval, *J. Research Nat. Bur. Stand.*, **45** (1950), 239—244.

[29] R. Vitale, A Bernstein polynomial approach to density estimation, in: *Statistical inference and related topics*, Academic Press (New York—San Francisco—London, 1975).

[30] H. Walk, *Approximation unbeschränkter Funktionen durch lineare positive Operatoren*, Habilitationsschrift, Stuttgart, 1970.

[31] H. Walk, Lokale Approximation unbeschränkter Funktionen und ihrer Ableitungen durch eine Klasse von Folgen linearer positiver Operatoren, *Mathematica*, **15** (1973), 129—142.

ABTEILUNG FÜR MATHEMATIK
UNIVERSITÄT ULM
OBERER ESELSBERG
7900 ULM (DONAU), FRG

Acta Sci. Math., 47 (1984), 223—231

О центрированных системах в $C[0, 1]$

А. В. БАХШЕЦЯН

Система измеримых функций $F=\{f_n\}$ называется центрированной, если для любого n и $\Lambda\in\mathscr{F}_n(F)$ ($\mathscr{F}_n(F)$ — минимальная σ-алгебра, по которой измеримы функции $f_1, f_2, \ldots, f_n$) имеет место

$$\int_\Lambda f_{n+1}(x)\,dx = 0.$$

Центрированные системы являются хорошо известным объектом теории вероятностей (точнее, теории мартингалов). Классическими примерами центрированных систем, определенных на отрезке [0, 1], являются системы Хаара и Радемахера.

Р. Ганди [1] доказал, что для полной в L^2 центрированной системы F σ-алгебра $\mathscr{F}_n(F)$ содержит ровно n различных атомов.

В работе [2] уточняется это утверждение, а именно, доказывается, что полная в L^2 [0, 1] нормированная центрированная система — с точностью до автоморфизма отрезка [0, 1] и перестановки, не нарушающей центрированность — совпадает с некоторой системой типа Хаара. Там же (см. [2]) строятся примеры нетривиальных неполных центрированных систем, состоящих из непрерывных функций и доказывается, что если $F=\{f_n\}$ бесконечная центрированная система п. в. отличных от нуля непрерывных на [0, 1] функций и некоторое f_{n_0} имеет ограниченную вариацию, то существует множество E с $\mu E=1$ (μ — мера Лебега) такое, что $\mu f_{n_0}(E)=0$. Далее строится контрпример, показывающий, что условие ограниченности вариации нельзя опустить. В данной работе мы усиливаем этот пример, доказывая следующее утверждение:

Теорема. *Пусть* $\{\varphi_n\}_{n=1}^\infty$ — *произвольная система непрерывных на* [0, 1] *функций, удовлетворяющих условию*

$$\int_0^1 \varphi_n(x)\,dx = 0$$

Поступило 12 сентября 1982 г.

для любого $n \geqq 2$. *Тогда существует центрированная система непрерывных функций* $\{f_n\}_{n=1}^{\infty}$ *такая, что* f_n *и* φ_n *равноизмеримы для любого* $n \geqq 1$.

Прежде чем перейти к доказательству теоремы, приведем некоторые вспомогательные замечания.

Отображение $T\colon [0, 1]\to[0, 1]$ называется эндоморфизмом, если оно сохраняет меру, т.е. для любого измеримого множества E множество $T^{-1}E$ также измеримо, причем $\mu T^{-1}E=\mu E$. Дополнительно мы потребуем непрерывность отображения T, так как в данной работе мы пользуемся лишь такими эндоморфизмами.

Замечание 1. Пусть $\{f_n(x)\}$ — центрированная система непрерывных функций и T — эндоморфизм. Тогда система $\{f_n(Tx)\}$ также является центрированной, причем для любого n функция $f_n(Tx)$ непрерывна и равноизмерима с $f_n(x)$.

Это сразу следует из соответствующих определений.

Замечание 2. Пусть последовательность эндоморфизмов $S_n(x)$ равномерно сходится к $S(x)$. Тогда $S(x)$ — эндоморфизм.

Действительно. Непрерывность $S(x)$ очевидна. Далее, для любого открытого множества U

$$S^{-1}U \subset \bigcup_{n=1}^{\infty}\bigcup_{k=n}^{\infty} S_k^{-1}U = \underline{\lim}\, S_k^{-1}U.$$

Обозначим через χ_A характеристическую функцию множества A. Из леммы Фату имеем

$$\mu(S^{-1}U) \leqq \mu(\underline{\lim}\, S_k^{-1}U) = \int_0^1 \chi_{\underline{\lim}\, S_k^{-1}U}\,dx = \int_0^1 \underline{\lim}\,\chi_{S_k^{-1}U}\,dx \leqq$$

$$\leqq \underline{\lim}\int_0^1 \chi_{S_k^{-1}U}\,dx = \underline{\lim}\,\mu(S_k^{-1}U) = \mu U.$$

Далее, для любого измеримого E и для любого $\varepsilon>0$ существует открытое множество U такое, что $E\subset U$, $\mu E>\mu U-\varepsilon$. Следовательно,

$$\mu^*(S^{-1}E) \leqq \mu(S^{-1}U) < \mu E+\varepsilon\,^{*)}$$

и в пределе $\mu^*(S^{-1}E)\leqq\mu E$. С другой стороны

$$1-\mu_*(S^{-1}E) = \mu^*(S^{-1}E^c) \leqq \mu E^c = 1-\mu E$$

($E^c=[0, 1]\setminus E$), т.е. $\mu_*(S^{-1}E)\geqq\mu E$. Этим заканчивается доказательство замечания 2.

---

*) μ^*, μ_* — соответственно, внешняя и внутренная меры Лебега.

Замечание 3. Пусть последовательность эндоморфизмов $\{T_i\}_{i=1}^{\infty}$ и числовая последовательность $\{b_i\}_{i=1}^{\infty}$ такие, что:

для любого n из $|x-y|<b_n$ следует $|S_n(x)-S_n(y)|<1/n$, где $S_n=T_1\circ T_2\circ\ldots\circ T_n$;

для любых n и $x\in[0, 1]$ $|T_{n+1}(x)-x|<\alpha_n$, причем $\sum_{i=n}^{\infty}\alpha_i<b_n$.

Тогда последовательность S_n равномерно сходится.

Действительно, при любых k и $x\in[0, 1]$

$$\left|T_{n+1}\left(T_{n+2}\left(\ldots T_{n+k}(x)\right)\right)-x\right| < \alpha_n+\alpha_{n+1}+\ldots+\alpha_{n+k-1} < b_n.$$

Следовательно,

$$|S_{n+k}(x)-S_n(x)| = \left|S_n\left(T_{n+1}\left(\ldots T_{n+k}(x)\right)\right)-S_n(x)\right| < 1/n.$$

Перейдем к доказательству теоремы.

Определение эндоморфизма S_Δ. Для каждого интервала $\delta=(\alpha, \beta)\subset\subset[0, 1]$ определим эндоморфизм S_δ° следующей формулой:

$$(1)\qquad S_\delta^\circ(x) = \begin{cases} x & \text{при } x\notin\delta; \\ 4x-3\alpha & \text{при } x\in(\alpha, (3\alpha+\beta)/4]; \\ 3\alpha+2\beta-4x & \text{при } x\in((3\alpha+\beta)/4, (\alpha+\beta)/2]; \\ 2x-\beta & \text{при } x\in((\alpha+\beta)/2, \beta). \end{cases}$$

Определим последовательность интервалов $\{\delta^i=(a_i, b_i)\}_1^\infty$ индуктивно: $\delta^1=\delta$;

$$\delta^{2n} = \left(\frac{a_n+b_n}{2}, \frac{a_n+3b_n}{4}\right); \quad \delta^{2n+1} = \left(\frac{a_n+3b_n}{4}, b_n\right).$$

Пусть $S_\delta^n=S_{\delta^1}^\circ\circ S_{\delta^2}^\circ\circ\ldots\circ S_{\delta^n}^\circ$. Докажем, что последовательность $\{S_\delta^n(x)\}$ сходится равномерно. Пусть $n\geqq k\geqq 2^{m+1}>1/\varepsilon$. Очевидно, каждое $x\in\delta^1$ покрывается конечной или бесконечной системой интервалов $\{\delta^{2^j+i_j}\}$ $(0\leqq i_j<2^j)$. Допустим

$$x\in\delta^1\cap\delta^{2+i_1}\cap\delta^{4+i_2}\cap\ldots.$$

Тогда, в силу (1), имеем

$$S_\delta^n(x) = S_{\delta^1}^\circ\circ S_{\delta^{2+i_1}}^\circ\circ\ldots\circ S_{\delta^{2^r+i_r}}^\circ(x),$$

$$S_\delta^k(x) = S_{\delta^1}^\circ\circ S_{\delta^{2+i_1}}^\circ\circ\ldots\circ S_{\delta^{2^p+i_p}}^\circ(x),$$

где либо $r=p$, либо $r>p\geqq m$. Очевидно, в первом случае $S_\delta^n(x)-S_\delta^k(x)=0$. Если же $r>p\geqq m$, то

$$|S_\delta^n(x)-S_\delta^k(x)| = 2^{p+1}|S_{\delta^{2^{p+1}+i_{p+1}}}^\circ\circ\ldots\circ S_{\delta^{2^r+i_r}}^\circ(x)-x| \leqq$$

$$\leqq 2^{p+1}\mu\delta^{2^{p+1}+i_{p+1}} \leqq 1/2^{p+1} \leqq 1/2^{m+1} < \varepsilon.$$

Мы здесь пользовались тем, что если $x, y \in \delta^{2q+i}$ $(i=0, 1)$, то

$$|S_\delta^q(x)-S_\delta^q(y)| = 2|x-y|.$$

Итак, в силу замечания 2, $\lim\limits_{n\to\infty} S_\delta^n = S_\delta$ — есть эндоморфизм.

Пусть $\Delta=\{\delta_i\}_{i=1}^\infty$ — последовательность непересекающихся интервалов из [0, 1] таких, что $\sum\limits_{i=1}^\infty \mu\delta_i=1$. (Далее такие последовательности будем называть разложениями отрезка [0, 1]). Очевидно, последовательность эндоморфизмов $\{S_{\delta_1}\circ S_{\delta_2}\circ\ldots\circ S_{\delta_n}\}_{n=1}^\infty$ равномерно сходится. Пусть

$$S_\Delta(x) = \lim_{n\to\infty} S_{\delta_1}\circ S_{\delta_2}\circ\ldots\circ S_{\delta_n}(x).$$

Докажем следующее свойство S_Δ:

(2) пересечение прообраза открытого множества с любым из

$$U_i^k = \delta_i^k \setminus (\bar{\delta}_i^{2k} \cup \delta_k^{2i+1}).$$

симметрично относительно середины интервала U_i^k.

Легко видеть, что $U_i^k \cap \delta_i^n = \emptyset$ при $n=k+1, k+2, \ldots$. Следовательно, на U_i^k

$$S_\Delta(x) = S_{\delta_i}(x) = S_{\delta_i}^k(x) = S_{\delta_i}^{k-1}\circ S^\circ_{\delta_i^k}(x).$$

Но, в силу (1), $S^\circ_{\delta_i^k}$ обладает указанным свойством.

Построение функции $\tilde{\varphi}(x)=\tilde{\varphi}(x, \Delta)$. Пусть φ непрерывная на [0, 1] функция, $\int\limits_0^1 \varphi(x)dx=0$ и $\Delta=\{\delta_i=(\alpha_i, \beta_i)\}_{i=1}^\infty$ — разложение [0, 1]. Мы хотим построить непрерывную на [0, 1] функцию $\tilde{\varphi}(x)=\tilde{\varphi}(x, \Delta)$, равноизмеримую с $\varphi(x)$ со следующими свойствами:

(3) $\int\limits_{\alpha_i}^{\beta_i} \tilde{\varphi}(x)dx=0$ для любого i;

(4) для любого i существует $\gamma_i \in (\alpha_i, \beta_i)$ такое, что $\tilde{\varphi}(x) \leqq 0$ при $\alpha_i \leqq x \leqq \gamma_i$, $\tilde{\varphi}(x) \geqq 0$ при $\gamma_i \leqq x \leqq \beta_i$;

(5) для любого i

$$\tilde{\varphi}(\gamma_i'-x) = \tilde{\varphi}(\gamma_i'+x) \quad \text{при} \quad |x| \leqq (\gamma_i-\alpha_i)/2$$

и

$$\tilde{\varphi}(\gamma_i''-x) = \tilde{\varphi}(\gamma_i''+x) \quad \text{при} \quad |x| \leqq (\beta_i-\gamma_i)/2,$$

где $\gamma_i'=(\alpha_i+\gamma_i)/2$, $\gamma_i''=(\gamma_i+\beta_i)/2$;

(6) на каждом из интервалов (α_i, γ_i'), (γ_i', γ_i'') и (γ_i'', β_i) $(i=1,2, \ldots)$ функция

$\tilde{\varphi}(x)$ монотонна;

(7) $$\frac{\mu(\{x\colon \tilde{\varphi}(x)=0\}\cap[\alpha_i, \gamma_i])}{\gamma_i-\alpha_i}=\frac{\mu(\{x\colon \tilde{\varphi}(x)=0\}\cap[\gamma_i, \beta_i])}{\beta_i-\gamma_i}.$$

Не нарушая общности, можем предположить, что функция φ неубывающая. Пусть γ_1 и $\sigma_1>0$ такие, что

$$\int_{\gamma_1}^{\gamma_1+\sigma_1}\varphi(x)\,dx=0, \qquad \max_{\gamma_1\leqq x\leqq\gamma_1+\sigma_1}|\varphi(x)|<M/2,$$

где $M=\|\varphi\|_c$. Существует натуральное n_1 такое, что $\sum_{i=1}^{n_1}(\beta_i-\alpha_i)>1-\sigma_1$. Обозначим $\sum_{i=1}^{n_1}(\beta_i-\alpha_i)=A_1$, $A_1-1+\sigma_1=a$.

Построим функцию ψ_1.

$$\psi_1(x)=\begin{cases}\varphi(x) & \text{при} \quad 0\leqq x<\gamma_1;\\ \varphi(\sigma_1(x-\gamma_1)/a+\gamma_1) & \text{при} \quad \gamma_1\leqq x<\gamma_1+a;\\ \varphi(x+1-A_1) & \text{при} \quad \gamma_1+a\leqq x<A_1;\\ \varphi(\sigma_1(x-A_1)/(1-A_1)+\gamma_1) & \text{при} \quad A_1\leqq x\leqq 1.\end{cases}$$

Следующие свойства функции ψ_1 очевидны:

(8) ψ_1 равноизмерима с φ;

(9) на интервалах $[0, A_1)$ и $[A_1, 1]$ ψ_1 непрерывна и неубывающая;

(10) $$\max_{A_1\leqq x\leqq 1}|\psi_1(x)|<M/2;$$

(11) $$\int_{A_1}^{1}\psi_1(x)\,dx=0.$$

Существуют $\gamma_2>A_1$ и $\sigma_2>0$ такие, что

$$\int_{\gamma_2}^{\gamma_2+\sigma_2}\psi_1(x)\,dx=0 \quad \text{и} \quad \max_{\gamma_2\leqq x\leqq\gamma_2+\sigma_2}|\psi_1(x)|\leqq M/3.$$

Выберем n_2 такое, что $\sum_{i=1}^{n_2}(\beta_i-\alpha_i)=A_2>1-\sigma_2$. Как и выше построим функцию ψ_2, удовлетворяющую условиям:

(12) $\psi_2(x)=\psi_1(x)$ на $[0, A_1)$;

(13) ψ_2 равноизмерима с φ;

(14) на интервалах $[0, A_1)$, $[A_1, A_2)$ и $[A_2, 1]$ ψ_2 непрерывна и неубывающая;

(15) $$\max_{A_2\leqq x\leqq 1}|\psi_2(x)|<M/3;$$

(16) $$\int_{A_2}^{1}\psi_2(x)\,dx=0.$$

Продолжая таким образом, мы получим последовательности $\{n_i\}_1^\infty$ и $\{\psi_i\}_1^\infty$ такие, что для любого i

(17) $$\psi_{i+1}(x)=\psi_i(x) \quad \text{на } [0, A_i), \quad \text{где} \quad A_i=\sum_{k=1}^{n_i}(\beta_k-\alpha_k);$$

(18) ψ_i равноизмерима с φ;

(19) на интервалах $[A_k, A_{k+1})$ $(k=0, \ldots, i-1;\ A_0=0)$ и $[A_i, 1]$ ψ_i непрерывна и неубывающая;

(20) $$\sup_{A_k\leqq x<A_{k+1}} |\psi_i(x)|<M/(k+1) \quad (k=0, 1, \ldots, i-1)$$

$$\text{и} \quad \max_{A_i\leqq x\leqq 1} |\psi_i(x)|<M/(i+1);$$

(21) $$\int_{A_k}^{A_{k+1}} \psi_i(x)\,dx=0, \quad k=0, 1, \ldots, i-1 \quad \text{и} \quad \int_{A_i}^{1} \psi_i(x)\,dx=0.$$

Из этих свойств сразу вытекает, что существует $\lim\limits_{i\to\infty}\psi_i(x)=\psi(x)$, причем ψ равноизмерима с φ, непрерывна при $x\neq A_i$ $(i=1, 2, \ldots)$, неубывающая на интервалах непрерывности и $\psi(1)=0$.

Обозначим

$$\bar{\bar{C}}_k'=\min\{x\colon x\geqq A_k, \psi(x)=0\}, \quad \bar{\bar{C}}_k''=\max\{x\colon x\leqq A_{k+1}, \psi(x)=0\}.$$

Пусть $A_{k+1}-c_k''=p_k(c_k'-A_k)$. Положим $c_k=(p_kc_k'+c_k'')/(p_k+1)$ и для $i=n_k+1$ $n_k+2, \ldots, n_{k+1}$; $\gamma_i=(p_k\alpha_i+\beta_i)/(p_k+1)$. Легко проверить, что функция

$$\tilde{\varphi}(x)=\begin{cases} \psi\big((A_k-c_k)(x-\alpha_i)/(\gamma_i'-\alpha_i)+c_k\big) & \text{при} \quad \alpha_i\leqq x\leqq \gamma_i';\\ \psi\big((A_{k+1}-A_k)(x-\gamma_i')/(\gamma_i''-\gamma_i')+A_k\big) & \text{при} \quad \gamma_i'<x\leqq\gamma_i'';\\ \psi\big((c_k-A_{k+1})(x-\gamma_i'')/(\beta_i-\gamma_i'')+A_{k+1}\big) & \text{при} \quad \gamma_i''<x\leqq\beta_i,\\ \qquad i=n_k+1, \ldots, n_{k+1}, \quad k=0, 1, \ldots \quad (n_0=0); & \\ 0 \quad \text{при} \quad x\in[0, 1]\diagdown\bigcup\limits_{i=1}^{\infty}\delta_i, & \end{cases}$$

где $\gamma_i'=(\alpha_i+\gamma_i)/2$, $\gamma_i''=(\gamma_i+\beta_i)/2$ — есть искомая.

Построение эндоморфизма $T_{\tilde{\varphi}}$.

Пусть $\Delta=\{\delta_i=(\alpha_i, \beta_i)\}_1^\infty$ — разложение $[0, 1]$ и функция $\tilde{\varphi}$ удовлетворяет условиям (3)—(7). Для каждого $x\in[\alpha_i, \beta_i]$ $(i=1, 2, \ldots)$ положим

$$\tilde{x}=\begin{cases}\beta_i-(\beta_i-\gamma_i)(x-\alpha_i)/(\gamma_i-\alpha_i) & \text{если } x\leqq\gamma_i \text{ и } \tilde{\varphi}(x)=0;\\ \alpha_i-(\gamma_i-\alpha_i)(x-\beta_i)/(\beta_i-\gamma_i) & \text{если } x>\gamma_i \text{ и } \tilde{\varphi}(x)=0.\end{cases}$$

Если же $\tilde{\varphi}(x)\neq 0$, то $\tilde{x}$ определим из условия

(22) $$\int_x^{\tilde{x}} \tilde{\varphi}(t)\,dt=0 \quad (\tilde{x}\in[\alpha_i, \beta_i]).$$

Отметим некоторые свойства отображения $x \to \tilde{x}$, выполняющиеся в силу (3)—(7).

(23) Равенство (22) выполняется при любом $x \in [\alpha_i, \beta_i]$ $(i=1, 2, \ldots)$.
(24) Отображение взаимно-однозначно.
(25) $\tilde{\varphi}(\tilde{x})=0$ тогда и только тогда, когда $\tilde{\varphi}(x)=0$.
(26) Для любого $x \in [\alpha_i, \beta_i]$, $\tilde{\tilde{x}}=x$.
(27) $\tilde{x}>x$ при $\alpha_i \leqq x < \gamma_i$, $\tilde{x}<x$ при $\gamma_i < x \leqq \beta_i$ и $\tilde{\gamma}_i=\gamma_i$.
(28) Если $\tilde{x}-x=\tilde{y}-y$, то $x=y$.

Это следует из того, что $\tilde{x}$ строго убывает при возрастании x.

Определим непрерывное отображение $T_{\tilde{\varphi}}$ следующей формулой:

$$T_{\tilde{\varphi}}(x) = \begin{cases} \beta_i - |\tilde{x}-x| & \text{при} \quad x \in \delta_i \quad (i = 1, 2, \ldots); \\ x & \text{при} \quad x \in [0, 1] \setminus \bigcup\limits_{i=1}^{\infty} \delta_i. \end{cases}$$

В силу (26), для любого $x \in \bigcup\limits_{i=1}^{\infty} \delta_i$ имеем $T_{\tilde{\varphi}}x=T_{\tilde{\varphi}}\tilde{x}$. Докажем, что $T_{\tilde{\varphi}}$ есть эндоморфизм. Действительно, пусть $0<\bar{C} \leqq (\beta_i-\alpha_i)/2$ и $x \in [\alpha_i, \gamma_i]$ такое, что $T_{\tilde{\varphi}}(x)=(\alpha_i+\beta_i)/2-c$. Тогда, в силу (28),

$$\begin{aligned} \mu T_{\tilde{\varphi}}^{-1}\big((\alpha_i+\beta_i)/2-c, (\alpha_i+\beta_i)/2\big) &= \mu\big((x, \gamma_i') \cup (\gamma_i'', \tilde{x})\big) = \\ &= \gamma_i'-x+\tilde{x}-\gamma_i'' = -(\gamma_i''-\gamma_i')+\beta_i-T_{\tilde{\varphi}}(x) = \\ &= -(\beta_i-\alpha_i)/2+\beta_i-(\alpha_i+\beta_i)/2+c = c. \end{aligned}$$

Аналогично, получим

$$\mu T_{\tilde{\varphi}}^{-1}\big((\alpha_i+\beta_i)/2, (\alpha_i+\beta_i)/2+c\big) = c$$

и, следовательно, для любого открытого $U \subset (\alpha_i, \beta_i)$, $\mu T_{\tilde{\varphi}}^{-1}(U)=\mu U$.

Отсюда рассуждениями, аналогичными приведенным при доказательстве замечания 2, получаем, что $T_{\tilde{\varphi}}$ есть эндоморфизм.

Легко также проверить, что если измеримое множество $E \subset [\alpha_i, \beta_i]$ симметрично относительно точки $(\alpha_i+\beta_i)/2$, то

$$\int\limits_{T_{\tilde{\varphi}}^{-1}(E)} \tilde{\varphi}(t)\, dt = 0. \tag{29}$$

Построение системы $\{f_n\}_{n=1}^{\infty}$. Положим $\Delta_1=\{(0, 1)\}$ и $\hat{f}_{11}=\varphi_1 \circ S_{\Delta_1}$. Пусть $\tilde{\Delta}_1=\{\tilde{\delta}_{i,1}\}_1^{\infty}$ — разбиение $[0, 1]$ такое, что

$$\hat{f}_{11}(c_{i,1}-x) = \hat{f}_{11}(c_{i,1}+x) \quad \text{при} \quad |x| < \mu\tilde{\delta}_{i,1}/2, \tag{30}$$

где $\bar{C}_{i,1}$ — середина интервала $\tilde{\delta}_{i,1}$ $(i=1, 2, \ldots)$. Существование $\tilde{\Delta}_1$ вытекает из (2).

Построим функцию $\tilde{\varphi}_2(x)=\tilde{\varphi}_2(x, \tilde{\Delta}_1)$ и эндоморфизм $T_{\tilde{\varphi}_2}$.

Положим $f_{11}=\hat{f}_{11}\circ T_{\tilde{\varphi}_2}$, $f_{21}=\tilde{\varphi}_2$. Очевидно, $f_{i,1}$ равноизмерима с φ_i $(i=1, 2)$.

В силу (30), для любого открытого множества пересечение $\tilde{\delta}_{i,1}\cap S_{\Delta_1}^{-1}\varphi_1^{-1}(U)$ симметрично относительно середины интервала $\tilde{\delta}_{i,1}$. Пусть $\tilde{\mathscr{F}}_{11}$ — минимальная σ-алгебра, содержащая множества вида $S_{\Delta_1}^{-1}\varphi_1^{-1}(U)$, а $\mathscr{F}_{11}$ — минимальная σ-алгебра, по которой измерима f_{11}. Очевидно, $\mathscr{F}_{11}=T_{\tilde{\varphi}_2}^{-1}\tilde{\mathscr{F}}_{11}$. Следовательно, для любого $\Lambda=T_{\tilde{\varphi}_2}^{-1}(\tilde{\Lambda})\in\mathscr{F}_{11}$, в силу (29),

$$\int_{\Lambda} f_{21}(t)\,dt = \int_{T_{\tilde{\varphi}_2}^{-1}(\tilde{\Lambda})} \tilde{\varphi}_2(t)\,dt = 0,$$

т.е. система из двух функций $\{f_{11}, f_{21}\}$ центрирована.

Далее, пусть $S_{\Delta_1}\circ T_{\tilde{\varphi}_2}=T_1$ и b_1 такое, что $|T_1(x)-T_1(y)|<1$ при $|x-y|<b_1$. Построим разбиение отрезка $[0, 1]$, $\Delta_2=\{\delta_{i,2}\}_{i=1}^{\infty}$, удовлетворяющую условию: $\max\limits_i \mu\delta_{i,2}<b_1/2$.

Обозначим: $\hat{f}_{k,2}=f_{k,1}\circ S_{\Delta_2}$ $(k=1, 2)$.

Пусть $\tilde{\Delta}_2=\{\tilde{\delta}_{i,2}\}_{i=1}^{\infty}$ — такое разбиение $[0, 1]$, что

$$\hat{f}_{k,2}(c_{i,2}-x) = \hat{f}_{k,2}(c_{i,2}+x) \quad \text{при} \quad |x| < \mu\tilde{\delta}_{i,2}/2 \quad (k = 1, 2).$$

Положим $f_{32}(x)=\tilde{\varphi}_3(x)=\tilde{\varphi}_3(x, \tilde{\Delta}_2)$, $f_{k,2}=\hat{f}_{k,2}\circ T_{\tilde{\varphi}_3}$ $(k=1, 2)$. Как и выше, легко показать что система $\{f_{k,2}\}_{k=1}^{3}$ — центрирована.

Продолжим таким образом, на каждом шаге требуя, чтобы для любого k выполнялось

$$\max_i \mu\delta_{i,k} < \min(b_1/2^{k-1}, b_2/2^{k-2}, \ldots, b_{k-1}/2), \tag{31}$$

где b_k такое, что из $|x-y|<b_k$ следует, что для $i=1, 2, \ldots, k$

$$|T_i\circ T_{i+1}\circ\ldots\circ T_k(x)-T_i\circ T_{i+1}\circ\ldots\circ T_k(y)| < 1/k \quad (T_m = S_{\Delta_m}\circ T_{\tilde{\varphi}_{m+1}}).$$

Получим последовательность функций $\{f_{k,i}\}_{i=k-1}^{\infty}$ $(k=1, 2, \ldots;\ f_{10}=\varphi_1)$ и эндоморфизмов $\{T_n\}_{n=1}^{\infty}$, удовлетворяющих условиям:

(32) $f_{k,i}$ равноизмерима с φ_k $(i=k-1, k, \ldots)$;
(33) для любого i система $\{f_{k,i}\}_{k=1}^{i+1}$ центрирована;
(34) $f_{k,i}=f_{k,i-1}\circ T_i$, $i=1, 2, \ldots;\ k=1, 2, \ldots, i$;
(35) для любого $x\in[0, 1]$ $|T_k(x)-x|<\max\limits_i \mu\delta_{i,k}$.

Из (35) и (31) следует, что любая последовательность эндоморфизмов $\{S_n^k=T_k\circ T_{k+1}\circ\ldots\circ T_n\}_{n=k}^{\infty}$ $(k=1, 2, \ldots)$ удовлетворяет условиям замечания 3 и, следовательно, сходится равномерно: $\lim\limits_{n\to\infty} S_n^k(x)=S^k(x)$.

Из (34) следует, что

$$\lim_{i\to\infty} f_{k,i}(x) = \lim_{i\to\infty} f_{k,m-1}(S_i^k(x)) = f_{k,m-1}(S^m(x)) \quad (m \geqq k).$$

Очевидно, предельная функция не зависит от m, так как для любого $n \geqq 1$, $T_n \circ T_{n+1} \circ \ldots \circ T_{m-1} \circ S^m = S^n$.

Обозначим: $f_{k,m-1}(S^m(x)) = f_k(x)$. Система $\{f_k\}_{k=1}^\infty$ — искомая. Действительно, для любого i

$$f_k(x) = f_{k,i-1}(S^i(x)), \quad k = 1, 2, \ldots, i.$$

Но S^i — эндоморфизм. Следовательно, в силу (32), (33) и замечания 1, f_k равноизмерима с φ_k и $\{f_k\}_{k=1}^i$ — центрирована.

Теорема полностью доказана.

Литература

[1] R. F. Gundy, Martingale theory and pointwise convergence of certain orthogonal series, *Trans. Amer. Math. Soc.*, **124** (1966), 228—248.

[2] А. В. Бахшецян, О структуре некоторых классов центрированных систем, *Изв. АН Арм. ССР. Сер. Мат.*, **13** (1978), 301—314.

ЕРЕВАНСКИЙ ГОСУДАРСТВЕННЫЙ УНИВЕРСИТЕТ
МЕХАНИКО-МАТЕМАТИЧЕСКИЙ ФАКУЛЬТЕТ
УЛ. МРАВЯНА, 1
375049 ЕРЕВАН, СССР

Acta Sci. Math., 47 (1984), 233—237

Small sum sets and the Faber gap condition

ROBERT E. DRESSLER and LOUIS PIGNO*)

Our purpose in this paper is to analyze sets satisfying a small sum set condition in terms of the classical Faber gap condition and to prove a "shape" result for sets of positive upper Banach density.

For simplicity's sake we will take all sequences (before Theorem 4) to be increasing sequences of non-negative integers. As usual, $\mathbf{Z}^+$ stands for the set of positive integers.

Definition 1 (The Faber gap condition of order p). If $S=\{s_j\}_{j=1}^{\infty}$ and $p\in\mathbf{Z}^+$, then S is said to be an F_p set if $s_{n+p}-s_n\to\infty$.

Definition 2 (Lacunarity condition L_n). A set S is said to satisfy condition L_1 if for every infinite sequence $\{n_j\}_{j=1}^{\infty}$, $|\underline{\lim}\,(S-n_j)|$ is finite. Proceeding inductively, we say that S satisfies condition L_n if for every sequence $\{n_j\}_{j=1}^{\infty}$ the set $\underline{\lim}\,(S-n_j)$ satisfies condition L_{n-1}.

Definition 3 (Generalized Faber gap condition $F_p^{(n)}$). If S is an L_n set that is not L_{n-1} and for some fixed p every n fold iterated lim inf has no more than p elements, we say S has property $F_p^{(n)}$.

Definition 4. If $n\geqq 2$, then a set S is said to be a ϱ_n set if

$$\sup\{\min(|A_1|, |A_2|, \ldots, |A_n|)\colon A_1+A_2+\ldots+A_n\subset S\}<\infty.$$

Examples of ϱ_n sets are constructed in [4] and [5]; see also [3]. Notice that any ϱ_n set $(n\geqq 2)$ is an L_{n-1} set.

Our first theorem shows that conditions F_p and $F_p^{(1)}$ coincide for all p.

Theorem 1. *S is an F_p set if and only if S is an $F_p^{(1)}$ set.*

Proof. Assume only for the sake of notation that $p\geqq 2$. We will show that $S=\{s_j\}_{j=1}^{\infty}$ is an F_p set but not an F_{p-1} set if and only if $\sup|\underline{\lim}\,(S-s_{j_i})|=p$, where the sup is taken over all subsequences $\{s_{j_i}\}_{i=1}^{\infty}\subset S$.

Received October 6, 1982 and in revised form March 10, 1983.

*) The second author was supported by NSF grant MCS-8102061.

First, suppose S is an F_p set, but not an F_{p-1} set. Then by the pigeon hole principle there is a subsequence $\{s_{j_i}\}_{i=1}^{\infty}$ such that for each t with $1 \leqq t \leqq p-1$, $s_{j_i+t}-s_{j_i}$ is constant for all i. It follows that

$$\{0, s_{j_1+1}-s_{j_1}, s_{j_1+2}-s_{j_1}, \ldots, s_{j_1+p-1}-s_{j_1}\}+\{s_{j_i}\}_{i=1}^{\infty} \subset S$$

which implies that $|\underline{\lim}\,(S-s_{j_i})| \geqq p$. On the other hand, if $|\underline{\lim}\,(S-s_{j_i})|>p$ for some $\{s_{j_i}\} \subset S$, then there exist $x_1<x_2<\ldots<x_{p+1}$ such that, for some $N \in \mathbf{Z}^+$ and each t with $1 \leqq t \leqq p+1$, $x_t+s_{j_i} \in S$ for all $i \geqq N$. Clearly then S is not an F_p set.

Now suppose $\sup |\underline{\lim}\,(S-s_{j_i})|=p$. If S were not an F_p set, then there would be $N \in \mathbf{Z}^+$ and a subsequence $\{s_{j_i}\}_{i=1}^{\infty} \subset S$ such that $s_{j_i+p}-s_{j_i}<N$ for all i. An argument similar to one we used above shows that $|\underline{\lim}\,(S-s_{j_i})| \geqq p+1$ which is a contradiction. In addition, if $\{s_{j_i}\}_{i=1}^{\infty} \subset S$ and $x_1<x_2<\ldots<x_p$ are such that for each t with $1 \leqq t \leqq p$, $x_t+s_{j_i} \in S$ for all sufficiently large i, then S is not an F_{p-1} set.

The next theorem relates the ϱ_2 property and the F_p property.

Theorem 2. *If S is a ϱ_2 set with* $\sup\{\min(|A_1|, |A_2|)\colon A_1+A_2 \subset S\}=p$, *then S is an F_p set.*

Proof. If S were not an F_p set, then there would exist $N \in \mathbf{Z}^+$ and a subsequence $\{s_{j_i}\}_{i=1}^{\infty} \subset S$ such that $s_{j_i+p}-s_{j_i}<N$ for all i. As in the proof of Theorem 1, there are infinitely many i's such that for each t with $1 \leqq t \leqq p$, $s_{j_i+t}-s_{j_i}$ is constant for all such i's. Thus,

$$\{0, s_{j_i+1}-s_{j_i}, s_{j_i+2}-s_{j_i}, \ldots, s_{j_i+p}-s_{j_i}\}+\{s_{j_i}\}_{i=1}^{\infty} \subset S$$

and so $\sup\{\min(|A_1|, |A_2|)\colon A_1+A_2 \subset S\} \geqq p+1$ which is a contradiction.

Next we relate the ϱ_n $(n \geqq 2)$ property and the iterative property $F_p^{(n)}$.

Theorem 3. *Let $n \geqq 3$. If S is a ϱ_n set with*

$$\sup\{\min(|A_1|, |A_2|, \ldots, |A_n|)\colon A_1+A_2+\ldots+A_n \subset S\} = p$$

and S is not an L_{n-2} set then S is an $F_p^{(n-1)}$ set.

Proof. Suppose S is as in the hypothesis and that, in contradiction of the conclusion, the cardinality of some $n-1$ fold iterated lim inf of S (with respect to sequences $\{s_{j_i}^{(k)}\}_{i=1}^{\infty}$, $0 \leqq k \leqq n-2$) is greater than p. Then there are $y_1<y_2<\ldots$ $\ldots<y_{p+1} \in \mathbf{Z}^+$ such that for each t with $1 \leqq t \leqq p+1$, $y_t+s_{j_i}^{(n-2)} \in \underline{\lim}\,\times$ $\times(\ldots(\underline{\lim}\,(\underline{\lim}\,(S-s_{j_i}^{(0)})-s_{j_i}^{(1)})\ldots-s_{j_i}^{(n-3)}))$ for all sufficiently large i. Next, for each t with $1 \leqq t \leqq p+1$, $y_t+s_{j_i}^{(n-2)}+s_{j_i}^{(n-3)} \in \underline{\lim}\,(\ldots(\underline{\lim}\,(S-s_{j_i}^{(0)})-s_{j_i}^{(1)})-\ldots-s_{j_i}^{(n-4)})$ for all sufficiently large i, etc.

Finally, we have, for each t with $1 \leqq t \leqq p+1$, $y_t+s_{j_i}^{(n-2)}+s_{j_i}^{(n-3)}+\ldots+s_{j_i}^{(1)}+$ $+s_{j_i}^{(0)} \in S$ for all sufficiently large i. This contradicts the hypothesis that S is ϱ_n and we are done.

Note. Theorem 3 tells us that if we have a ϱ_n set that is not L_{n-2} then whenever we take $n-2$ lim infs we have arrived at either an F_p set or a finite set.

The next theorem, of interest in itself, and whose proof may remind the reader of Lukomskaya's proof of van der Waerden's Theorem in [6], will help us relate density to the $F_p^{(n)}$ property. We need the following definitions.

Definition 5. If $E\subset \mathbf{Z}$, then the upper Banach density of E is defined by $\overline{Bd}(E)=\limsup_{|I|\to\infty}(|E\cap I|)/|I|$, where I ranges over all bounded intervals in $\mathbf{Z}$.

Theorem 4. *Let* $E=\{e_i\}_{i=1}^{\infty}$. *If* $\overline{Bd}(E)>0$ *then for each* $j\geqq 0$, *there exists* e_{i_j}, M_j *and* k_j *such that*

$$\{e_{i_j}, \dots, e_{i_j+M_j}\}+k_j\subset\{e_{i_{j+1}}, \dots, e_{i_{j+1}+M_{j+1}}\}$$

with $0<M_j<M_{j+1}$, *and, for* $j_1\neq j_2$,

$$\{e_{i_{j_1}}, \dots, e_{i_{j_1}+M_{j_1}}\}\cap\{e_{i_{j_2}}, \dots, e_{i_{j_2}+M_{j_2}}\}=\emptyset.$$

In addition, there exists $M>0$ *such that for each* j,

$$M_j+1>\left|\frac{e_{i_j+M_j}-e_{i_j}}{M}\right|.$$

Proof. Say $\overline{Bd}(E)>1/N_0$ for some $N_0\in\mathbf{Z}^+$. Then there exist infinitely many integers x_0 such that $|E\cap[x_0, x_0+N_0]|>N_0/2N_0=1/2$. Form at most $2^{N_0+1}-1$ classes of such intervals $[x_0, x_0+N_0]$ according to the "shape" of $E\cap[x_0, x_0+N_0]$. Call such a class by the generic name C. At least one C_0 is infinite.

Next, choose $N_1\in\mathbf{Z}^+$ such that N_0 divides N_1 and $N_1/2N_0>N_0+1$. Now there exist infinitely many integers x_1 such that $E\cap[x_1, x_1+N_1]>N_1/2N_0$. We take such intervals that contain a member of some class C_0. Notice that by breaking up intervals of length N_1 into consecutive intervals of length N_0, we see that, except for finitely many intervals, each interval of length N_1 which has at least $N_1/2N_0$ members of E must contain at least one member of one class C_0 (for if not, all but finitely many such intervals of length N_1 would have fewer than $(N_1/N_0)(N_0/2N_0)=N_1/2N_0$ members). Since there exist infinitely many intervals of length N_1 which contain a member of some C_1 we must have infinitely many such intervals containing infinitely many members of the same class C_0. In addition, if we classify these intervals $[x_1, x_1+N_1]$ according to the "shape" of $E\cap[x_1, x_1+N_1]$ we see that at least one such class of intervals is infinite. Call such classes of intervals of length N_1 by the generic name C_1. Notice that since $N_1/2N_0>N_0+1$, we see that the intersection of E with any member of any C_1 has more elements than the intersection of E with any member of any C_0.

Now choose $N_2\in\mathbf{Z}^+$ such that N_1 divides N_2 and $N_2/2N_0>N_1+1$. We repeat the construction to obtain at least one infinite class C_2 of intervals of the form

$[x_2, x_2+N_2]$ all of whose members have the same "shape" when intersected with E and, for some class C_1, each of whose members contains a member of C_1. Continue in this manner. Notice that at the j^{th} stage, starting with $j=0$, only finitely many of the intervals $[x_j, x_j+N_j]$ which contain at least a proportion $1/2N_j$ of elements of E do not belong to some C_j. Also, no two classes C_j need be disjoint.

We thus may obtain, for each $j \geqq 0$, an interval $[x_j, x_j+N_j] \in C_j$ such that the intervals are pairwise disjoint and such that for each $j \geqq 0$ there is a $k_j \in \mathbf{Z}$ with

$$(E \cap [x_j, x_j+N_j]) + k_j \subset (E \cap [x_{j+1}, x_{j+1}+N_{j+1}])$$

where since $N_{j+1}/2N_0 > N_j+1$,

$$|E \cap [x_j, x_j+N_j]| < |E \cap [x_{j+1}, x_{j+1}+N_{j+1}]|.$$

If we write $\{e_{i_j}, \ldots, e_{i_j+M_j}\} = E \cap [x_j, x_j+N_j]$, the proof is complete.

Corollary. *If E is an L_n set for some n then $\overline{Bd}(E)=0$.*

Proof. If $\overline{Bd}(E)>0$, then if e_0 is the smallest element of $E \cap [x_0, x_0+N_0]$ and $\{n_j\}_{j=0}^{\infty} = \{e_0, e_0+k_0, e_0+k_0+k_1, \ldots\}$ then $\underline{\lim}\,(E-n_j)$ contains the set

$$B_1 = \{0, e_2-e_1, e_3-e_1, \ldots, e_{r_0}-e_1, e'_{r+1}-e'_1, \ldots$$
$$\ldots, e'_{r_1}-e'_1, e^{(2)}_{r+1}-e^{(2)}_1, \ldots, e^{(2)}_{r_2}-e^{(2)}_1, \ldots\}$$

where

$$E \cap [x_0, x_0+N_0] = \{e_1, \ldots, e_{r_0}\},$$
$$E \cap [x_1, x_1+N_1] = \{e'_1, \ldots, e'_{r_1}\},$$
$$\vdots$$
$$E \cap [x_j, x_j+N_j] = \{e^{(j)}_1, \ldots, e^{(j)}_{r_j}\},$$
$$\vdots$$

By construction $\overline{Bd}(B_1)>0$ because for each j, $\{e^{(j)}_1, \ldots, e^{(j)}_{r_j}\}$ contains at least a proportion $1/2N_0$ of the interval $[x_j, x_j+N_j]$.

Now, perform the construction another $n-1$ times to obtain a set B_n with $\overline{Bd}(B_n)>0$. Clearly then E is not an L_n set.

Definition 6. Let $N \in \mathbf{Z}^+$; a subset P of $\mathbf{Z}$ is a parallelepiped of dimension N if P has exactly 2^N elements and can be represented as a sum $P_1+\ldots+P_N$ of N two-element subsets.

It is easy to see that if E does not contain parallelepipeds of arbitrarily large dimension, then E is an L_n set for some n. Thus, an easy consequence of our corollary is that if $\overline{Bd}(E)>0$, then E contains parallelepipeds of arbitrarily large dimension; this fact for natural density was first pointed out in [1]. It is also proved in [1] that if E is a p-Sidon set, a $\Lambda(1)$ set, or a UC-set, then there is a $N \in \mathbf{Z}^+$ for which E contains no parallelepiped of dimension N; see [1] and [4] for some definitions. It is also known that such an analytically defined E cannot contain arithmetic progressions of arbitrary length, hence, it also follows from the work of E. Szemerédi [7] that such sets have natural density zero. On the other hand, it is quite easy to construct an L_n set containing arbitrarily long arithmetic progressions.

We conclude our paper with two examples which delineate the scope of Theorem 3.

Example 1. Let $A_1=\{3^{3^n}+3^{2^1}\colon n\in\mathbf{Z}^+\}$, $A_2=\{3^{5^n}+3^{2^2}\colon n\in\mathbf{Z}^+\}\cup\{3^{5^n}+3^{2^3}\colon n\in\mathbf{Z}^+\}$, ..., $A_t=\{3^{p_t^n}+3^{2^{t'}}\colon n\in\mathbf{Z}^+\}\cup\dots\cup\{3^{p_t^n}+3^{2^{t'+t-1}}\colon n\in\mathbf{Z}^+\}$ where p_t is the t^{th} prime and $t'=1+\sum_{i=1}^{t-1} i$. Let $S=\bigcup_{t=1}^{\infty} A_t$. Then S is clearly not ϱ_2 but since $S\subset\{3^{3^n}\colon n\in\mathbf{Z}^+\}+\{3^{2^n}\colon n\in\mathbf{Z}^+\}$, S is a ϱ_3 set.

In addition, it follows from [2, p. 76] that S is an L_1 set that is not F_p for any p. An appeal to Theorem 1 now confirms the following assertion: Given any natural number p there is a sequence $\{n_j\}_{j=1}^{\infty}$ such that $\infty>|\underline{\lim}\,(S-n_j)|>p$.

Example 2. Fix $q\in\mathbf{Z}^+$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be pairwise disjoint increasing sequences of positive integers. Let $A_1^1=\{3^{b_1}\}$ and $A_2^1=\{3^{c_1}\}$. Let $A_1^2=\{3^{b_2}, 3^{b_3}\}$ and $A_2^2=\{3^{c_2}, 3^{c_3}\}$. Let $A_1^3=\{3^{b_4}, 3^{b_5}, 3^{b_6}\}$ and $A_2^3=\{3^{c_4}, 3^{c_5}, 3^{c_6}\}$, etc. Finally, let

$$S=\bigcup_{n=1}^{\infty}[(\{3^{a_1}, 3^{a_2}, \dots, 3^{a_q}\}\cup A_1^n)+A_2^n].$$

By construction, S is L_1 (indeed it is F_{q+1}). Also if $\{d_n\}$ is any sequence such that $d_n\in A_2^n$ for each n, then $\{3^{a_1}, 3^{a_2}, \dots, 3^{a_q}\}\subset\underline{\lim}\,(S-d_n)$. It follows from say [4] that

$$\sup\{\min(|A_1|, |A_2|, |A_3|)\colon A_1+A_2+A_3\subset S\}$$

is bounded above for all q. Clearly q could have been chosen greater than this bound; thus $\sup\{\min(|A_1|, |A_2|, |A_3|)\colon A_1+A_2+A_3\subset S\}<\sup|\underline{\lim}\,(S-n_j)|=q+1$, for q sufficiently large.

Examples 1 and 2 show that the hypothesis in Theorem 3 that S not be an L_{n-2} set cannot be relaxed.

References

[1] J. J. F. Fournier and L. Pigno, Analytic and arithmetic properties of thin sets, *Pacific J. Math.*, **105** (1983), 115—141.

[2] R. E. Dressler and L. Pigno, Some lacunarity conditions for Fourier—Stieltjes transforms, *Ark. Math.*, **13** (1975), 73—77.

[3] R. E. Dressler, On sets which contain sum sets, *Acta Sci. Math.*, **42** (1980), 233—235.

[4] R. E. Edwards and K. A. Ross, *p*-Sidon sets, *J. Funct. Anal.*, **15** (1974), 404—427.

[5] G. W. Johnson and G. S. Woodward, On *p*-Sidon sets, *Indiana J. Math.*, **24** (1974), 161—168.

[6] A. Y. Khinchin, *Three pearls of number theory*, Graylock Press (Rochester, N. Y., 1952).

[7] E. Szemerédi, On sets of integers containing no *k* elements in arithmetic progression, *Acta Arith.*, **27** (1975), 199—245.

DEPARTMENT OF MATHEMATICS
KANSAS STATE UNIVERSITY
MANHATTAN, KANSAS 66 506, U.S.A.

Acta Sci. Math., 47 (1984), 239—247

On approximation of the solutions of quasi-linear elliptic equations in $\mathbf{R}^n$

L. SIMON

Introduction

Let $P=P(D)$ be an elliptic differential operator of order $2m$ with constant coefficients $\left(D=\left(-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_n}\right)\right)$ and $Q=Q(x, D)$ a differential operator of order $2m$ with smooth coefficients which vanish for $|x|>a$.

For any domain $\Omega\subset\mathbf{R}^n$ and any integer $k\geqq 0$ denote by $H^k(\Omega)$ the Hilbert space of functions u with the norm

$$\|u\|_{H^k(\Omega)}=\Big\{\sum_{|\alpha|\leqq k}\int_\Omega |D^\alpha u|^2\,dx\Big\}^{1/2}$$

(Sobolev space); $L^2(\Omega)=H^0(\Omega)$. Further denote by $H^k_{\mathrm{loc}}(\Omega)$ the set of functions u satisfying the condition: $\varphi u\in H^k(\Omega)$ for arbitrary infinitely differentiable function φ which is equal to zero out of a compact subset of Ω.

In [1] the elliptic equation

$$Au\equiv(P+Q)u=f\quad\text{in}\quad\mathbf{R}^n \tag{0.1}$$

has been considered when $P(\xi)\neq 0$ for all $\xi\in\mathbf{R}^n$. It has been proved that if for any $f\in L^2_a(\mathbf{R}^n)$ $\big($i.e. $f\in L^2(\mathbf{R}^n)$, $f(x)=0$ if $|x|>a\big)$ there exists a solution u of the equation (0.1) which tends to zero at infinity then the solution is unique. Furthermore, by use of methods of [2] it is easy to show that for this solution the estimation

$$\|u\|_{H^{2m}(\mathbf{R}^n)}\leqq c_1\|f\|_{L^2_a(\mathbf{R}^n)} \tag{0.2}$$

holds. (c_1 is a constant which does not depend on f.) In [1] there have been formulated conditions on the differential operators $B_j(x, D)$ which guarantee that for sufficiently

Received October 20, 1982.

large $\varrho>0$ the boundary value problem in $B_\varrho=\{x\in\mathbf{R}^n:\ |x|<\varrho\}$

$$Au_\varrho=f \quad \text{in} \quad B_\varrho \tag{0.3}$$

$$B_j(x, D)u_\varrho=0 \quad \text{on} \quad S_\varrho, \quad j=1, \ldots, m \tag{0.4}$$

($S_\varrho=\{x\in\mathbf{R}^n:\ |x|=\varrho\}$) has a unique solution u_ϱ in the Sobolev space $H^{2m}(B_\varrho)$ and an estimation of the form

$$\|u-u_\varrho\|_{H^{2m}(B_\varrho)}\leqq c_2\|f\|_{L^2_a(\mathbf{R}^n)}e^{-c_3\varrho} \tag{0.5}$$

holds, where c_2, c_3 are positive constants which do not depend on f and ϱ.

In [3] similar results are proved when $P(\xi)\neq 0$ for $\xi\in\mathbf{R}^n\setminus\{0\}$ but $P(0)=0$ and (0.3), (0.4) is the Dirichlet problem, i.e. $B_j(x, D)=\dfrac{\partial^j}{\partial\nu^j}$ where ν is the normal vector to S_ϱ. Then instead of (0.2) and (0.5) the following estimations are valid: for any compact $K\subset R^n$

$$\|u\|_{H^{2m}(K)}\leqq c_1(K)\|f\|_{L^2_a(\mathbf{R}^n)} \tag{0.6}$$

and

$$\|u-u_\varrho\|_{H^{2m}(K)}\leqq c_2(K)g(\varrho)\|f\|_{L^2_a(\mathbf{R}^n)}, \tag{0.7}$$

where $c_1(K), c_2(K)$ are constants which do not depend on f and ϱ, $\lim\limits_{\varrho\to+\infty}g(\varrho)=0$. Under certain conditions estimations of the form (0.2), resp.

$$\|u-u_\varrho\|_{H^{2m}(\mathbf{R}^n)}\leqq g(\varrho)\|f\|_{L^2_a(\mathbf{R}^n)} \tag{0.8}$$

can be shown where $\lim\limits_{\varrho\to+\infty}g(\varrho)=0$.

In this paper it will be supposed that the differential operators P and Q satisfy the above mentioned conditions of works [1], resp. [3] such that estimations of the form (0.6), (0.7), resp. (0.2), (0.8) hold. Our aim is to consider a quasi-linear elliptic equation of the form

$$Au+g(x, u, \ldots, D^\beta u, \ldots)=f \quad \text{in} \quad \mathbf{R}^n \tag{0.9}$$

where $|\beta|\leqq 2m-1$ and to prove the existence of a solution of (0.9). Moreover, we are going to prove an estimation of type (0.7), resp. (0.8) for the quasi-linear equation (0.9).

In [4]—[8] there are proved existence theorems on quasi-linear and nonlinear elliptic equations in unbounded domains. These results, however, cannot be applied to the equation (0.9) in the case $P(0)=0$.

1. Existence of solutions

Theorem 1. *Suppose that for any* $f \in L_a^2(\mathbf{R}^n)$ *there exists a unique solution* u *of* (0.1) *and for this solution the estimation* (0.6) *holds. Let* $g\colon \mathbf{R}^{n+N} \to \mathbf{R}$ *be a continuous function* (*N denotes the number of multiindices* β *such that* $|\beta| \leqq 2m-1$) *satisfying the conditions:*

$$g(x, u, \ldots, u_\beta, \ldots) = 0 \quad \text{if} \quad |x| > a; \tag{1.1}$$

$$\lim_{|(u, \ldots, u_\beta, \ldots)| \to \infty} \frac{g(x, u, \ldots, u_\beta, \ldots)}{|(u, \ldots, u_\beta, \ldots)|} = 0 \quad \text{uniformly in } x; \tag{1.2}$$

(1.3) *the first partial dervatives of g are continuous and bounded.*

Then for any $f \in L_a^2(\mathbf{R}^n)$ *equation* (0.9) *has at least one solution* $u \in H^{2m}_{\mathrm{loc}}(\mathbf{R}^n)$, *vanishing at infinity.*

Proof. Denote by $A^{-1}f$ the unique solution of (0.1) which vanishes at infinity. Function u is the solution of (0.9) (vanishing at infinity) if and only if $v = Au$ is a solution of the equation

$$v + G(v) = f \tag{1.4}$$

in $L_a^2(\mathbf{R}^n)$ where the operator G is defined by

$$G(v) = g(x, A^{-1}v, \ldots, D^\beta A^{-1}v, \ldots).$$

We shall first prove that G is a continuous and compact (nonlinear) operator in the Hilbert space $L_a^2(\mathbf{R}^n)$. By use of the mean value theorem and condition (1.3) we have the estimation

$$|G(v) - G(v^*)| \leqq c_1[|A^{-1}(v-v^*)| + \ldots + |D^\beta A^{-1}(v-v^*)| + \ldots]$$

(c_1 denotes a constant) and thus

$$\Big\{ \int_{B_a} |G(v) - G(v^*)|^2 \Big\}^{1/2} \leqq \tag{1.5}$$

$$\leqq c_1 \Big[\Big\{ \int_{B_a} |A^{-1}(v-v^*)|^2 \Big\}^{1/2} + \ldots + \Big\{ \int_{B_a} |D^\beta A^{-1}(v-v^*)|^2 \Big\}^{1/2} + \ldots \Big] \leqq c_2 \|v - v^*\|_{L_a^2(\mathbf{R}^n)},$$

because in virtue of (0.6) $A^{-1}\colon L_a^2(\mathbf{R}^n) \to H^{2m}(B_a)$ is a bounded linear operator. Since A^{-1} is a bounded linear operator and by (1.2)

$$|g(x, u, \ldots, u_\beta, \ldots)| \leqq c_3 |(u, \ldots, u_\beta, \ldots)|$$

(c_3 denotes a constant), thus

$$\|G(v)\|_{L^2(B_a)} \leqq c_4 \|A^{-1}v\|_{H^{2m}(B_a)} \leqq c_5 \|v\|_{L_a^2(\mathbf{R}^n)}.$$

Hence by use of condition (1.1) we find that for any $v \in L_a^2(\mathbf{R}^n)$, $G(v) \in L_a^2(\mathbf{R}^n)$ and thus by (1.5) $G\colon L_a^2(\mathbf{R}^n) \to L_a^2(\mathbf{R}^n)$ is a continuous operator.

From conditions (1.1)—(1.3) it follows that for any $v \in L_a^2(\mathbf{R}^n)$

$$\frac{\partial}{\partial x_j} G(v) = \frac{\partial g}{\partial x_j}(x, A^{-1}v, \ldots; D^\beta A^{-1}v, \ldots) + \tag{1.6}$$

$$+ \sum_{|\beta| \leqq 2m-1} \frac{\partial g}{\partial u_\beta}(x, A^{-1}v, \ldots, D^\beta A^{-1}v, \ldots) \frac{\partial}{\partial x_j}(D^\beta A^{-1}v)$$

and $G(v) \in H_a^1(\mathbf{R}^n)$ (i.e. $G(v) \in H^1(\mathbf{R}^n)$ and $G(v)=0$ for $|x|>a$). From (1.6) it is also clear that G maps bounded subsets of $L_a^2(\mathbf{R}^n)$ into bounded subsets of $H_a^1(\mathbf{R}^n)$. Hence $G\colon L_a^2(\mathbf{R}^n) \to L_a^2(\mathbf{R}^n)$ is a compact operator.

Now we shall prove the equality

$$\lim_{\|v\|_{L_a^2(\mathbf{R}^n)} \to \infty} \frac{\|G(v)\|_{L_a^2(\mathbf{R}^n)}}{\|v\|_{L_a^2(\mathbf{R}^n)}} = 0. \tag{1.7}$$

Denote $A^{-1}v$ by u then

$$\frac{\|G(v)\|_{L_a^2(\mathbf{R}^n)}}{\|v\|_{L_a^2(\mathbf{R}^n)}} = \frac{\|g(x, u, \ldots, D^\beta u, \ldots)\|_{L_a^2(\mathbf{R}^n)}}{\|u\|_{H^{2m}(B_a)}} \cdot \frac{\|u\|_{H^{2m}(B_a)}}{\|v\|_{L_a^2(\mathbf{R}^n)}}. \tag{1.8}$$

In virtue of the boundedness of A^{-1} the second factor on the right hand side is bounded. Moreover, $\|u\|_{H^{2m}(B_a)} \to \infty$ as $\|v\|_{L_a^2(\mathbf{R}^n)} \to \infty$ since $v = A(u)$ and $A\colon H^{2m}(B_a) \to L^2(B_a)$ is a bounded linear operator. Thus to prove (1.7) we have only to show that

$$\lim_{\|u\|_{H^{2m}(B_a)} \to \infty} \frac{\|g(x, u, \ldots; D^\beta u, \ldots)\|_{L_a^2(\mathbf{R}^n)}}{\|u\|_{H^{2m}(B_a)}} = 0. \tag{1.9}$$

For any positive number $b>0$

$$\int_{B_a} |g(x, u(x), \ldots, D^\beta u(x), \ldots)|^2 \, dx = \tag{1.10}$$

$$= \int_{|(u(x), \ldots, D^\beta u(x), \ldots)| > b} |g(x, u(x), \ldots, D^\beta u(x), \ldots)|^2 \, dx +$$

$$+ \int_{|(u(x), \ldots, D^\beta u(x), \ldots)| \leqq b} |g(x, u(x), \ldots, D^\beta u(x), \ldots)|^2 \, dx.$$

By (1.2) for any $\varepsilon > 0$ the number $b>0$ can be chosen such that

$$|g(x, u(x), \ldots, D^\beta u(x), \ldots)| \leqq \varepsilon |(u(x), \ldots, D^\beta u(x), \ldots)|.$$

Thus

$$\text{(1.11)} \qquad \int_{|(u(x),\ldots,D^\beta u(x),\ldots)|>b} |g(x,u(x),\ldots;D^\beta u(x),\ldots)|^2\,dx \leqq$$

$$\leqq \varepsilon^2 \int_{B_a} |(u(x),\ldots,D^\beta u(x),\ldots)|^2\,dx \leqq \varepsilon^2 \|u\|^2_{H^{2m}(B_a)}.$$

For a fixed $b>0$ the second term on the right in (1.10) is bounded because g is continuous and $|x|\leqq a$, $|(u(x),\ldots,D^\beta u(x),\ldots)|\leqq b$. Therefore from (1.10), (1.11) we have (1.9) and equality (1.7) is proved.

Since $G\colon L^2_a(\mathbf{R}^n)\to L^2_a(\mathbf{R}^n)$ is a continuous compact operator satisfying (1.7), thus by use of Schauder's fixed point theorem we can prove that the equation (1.4) has at least one solution $v\in L^2_a(\mathbf{R}^n)$ for any $f\in L^2_a(\mathbf{R})$. By (1.7) we can choose a number $\varrho_0>0$ such that

$$\|v\|_{L^2_a(\mathbf{R}^n)}>\varrho_0 \quad \text{implies} \quad \frac{\|G(v)\|_{L^2_a(\mathbf{R}^n)}}{\|v\|_{L^2_a(\mathbf{R}^n)}}<\frac{1}{2}.$$

Set $F(v)=f-G(v)$. Then the operator F is bounded in $L^2_a(\mathbf{R}^n)$, i.e.

$$\|v\|_{L^2_a(\mathbf{R}^n)}\leqq\varrho_0 \quad \text{implies} \quad \|F(v)\|_{L^2_a(\mathbf{R}^n)}\leqq\varrho_1,$$

since G is bounded in $L^2_a(\mathbf{R}^n)$. Let ϱ denote the number $\max\{\varrho_0,\varrho_1,2\|f\|\}$. Then F maps the sphere $\{v\in L^2_a(\mathbf{R}^n)\colon \|v\|_{L^2_a(\mathbf{R}^n)}\leqq\varrho\}$ into itself, because $\|F(v)\|\leqq\varrho_1\leqq\varrho$ if $\|v\|\leqq\varrho_0$ and

$$\|F(v)\|\leqq\|f\|+\|G(v)\|\leqq\varrho/2+\|v\|/2\leqq\varrho \quad \text{if} \quad \varrho_0\leqq\|v\|\leqq\varrho.$$

Moreover, F is a continuous and compact operator, hence by Schauder's fixed point theorem F has at least one fixed point. Thus equation (1.4) has at least one solution $v\in L^2_a(\mathbf{R}^n)$ and then the function $u=A^{-1}v\in H^{2m}_{\mathrm{loc}}(\mathbf{R}^n)$ is a solution of (0.1), vanishing at infinity.

Consider now the following boundary value problem in B_ϱ:

$$\text{(1.12)} \qquad Au_\varrho+g(x,u_\varrho,\ldots;D^\beta u_\varrho,\ldots)=f \quad \text{in} \quad B_\varrho,$$

$$\text{(1.13)} \qquad B_j(x,D)u_\varrho=0 \quad \text{on} \quad S_\varrho,\ j=1,\ldots,m.$$

Theorem 2. *Assume that the conditions of Theorem 1 are fulfilled. Further suppose that if* $\varrho\geqq\varrho_0$ *then for any* $f\in L^2_a(\mathbf{R}^n)$ *the problem* (0.3), (0.4) *has a unique solution* $u_\varrho\in H^{2m}(B_\varrho)$ *and the estimation* (0.7) *holds. Then for any* $\varrho\geqq\varrho_0$ *and* $f\in L^2_a(\mathbf{R}^n)$ *the problem* (1.12), (1.13) *has at least one solution* $u_\varrho\in H^{2m}(B_\varrho)$.

Proof. Denote by $A_\varrho^{-1}f$ the unique solution $u_\varrho\in H^{2m}(B_\varrho)$ of the problem (0.3), (0.4). If $v_\varrho\in L^2_a(\mathbf{R}^n)$ is a solution of

$$\text{(1.14)} \qquad v_\varrho+g(x,A_\varrho^{-1}v_\varrho,\ldots,D^\beta A_\varrho^{-1}v_\varrho,\ldots)=f$$

then $u_\varrho = A_\varrho^{-1} v_\varrho \in H^{2m}(B_\varrho)$ is a solution of (1.12), (1.13). Define an operator G_ϱ by the formula

$$G_\varrho(v_\varrho) = g(x, A_\varrho^{-1} v_\varrho, \ldots, D^\beta A_\varrho^{-1} v_\varrho, \ldots).$$

Then $G_\varrho \colon L_a^2(\mathbf{R}^n) \to L_a^2(\mathbf{R}^n)$ is a continuous and compact operator and

$$(1.15) \qquad \lim_{\|v\|_{L_a^2(\mathbf{R}^n)} \to \infty} \frac{\|G_\varrho(v)\|_{L_a^2(\mathbf{R}^n)}}{\|v\|_{L_a^2(\mathbf{R}^n)}} = 0 \quad \text{uniformly for } \varrho \geqq \varrho_0.$$

This statement can be verified by means analogous to those used before in proving Theorem 1. We want only to show the proof of (1.15). Since

$$A_\varrho^{-1} v = (A_\varrho^{-1} - A^{-1}) v + A^{-1} v,$$

thus by estimations (0.6) and (0.7) $A_\varrho^{-1} \colon L_a^2(\mathbf{R}^n) \to H^{2m}(B_a)$ is a bounded linear operator and $\|A_\varrho^{-1}\|$ is uniformly bounded for $\varrho \geqq \varrho_0$:

$$(1.16) \qquad \frac{\|A_\varrho^{-1} v\|_{H^{2m}(B_a)}}{\|v\|_{L_a^2(\mathbf{R}^n)}} \leqq c_1$$

for any $v \in L_a^2(\mathbf{R}^n)$ and $\varrho \geqq \varrho_0$. Further

$$(1.17) \qquad \|A_\varrho^{-1} v\|_{H^{2m}(B_a)} \to \infty \quad \text{uniformly for} \quad \varrho \geqq \varrho_0 \quad \text{as} \quad \|v\|_{L_a^2(\mathbf{R}^n)} \to \infty,$$

since $v = A(A_\varrho^{-1} v)$ and $A \colon H^{2m}(B_a) \to L^2(B_a)$ is a bounded linear operator which does not depend on ϱ. The equality

$$\frac{\|G_\varrho(v)\|_{L_a^2(\mathbf{R}^n)}}{\|v\|_{L_a^2(\mathbf{R}^n)}} = \frac{\|g(x, A_\varrho^{-1} v, \ldots, D^\beta A_\varrho^{-1} v, \ldots)\|_{L_a^2(\mathbf{R}^n)}}{\|A_\varrho^{-1} v\|_{H^{2m}(B_a)}} \cdot \frac{\|A_\varrho^{-1} v\|_{H^{2m}(B_a)}}{\|v\|_{L_a^2(\mathbf{R}^n)}}$$

and (1.9), (1.16), (1.17) imply (1.15).

Thus by use of Schauder's fixed point theorem we find that there exists a solution v_ϱ of (1.14) (see the proof of Theorem 1), hence $u_\varrho = A_\varrho^{-1} v_\varrho \in H^{2m}(B_\varrho)$ is a solution of (1.12), (1.13).

2. Theorem on approximation

Theorem 3. *Suppose that all conditions of Theorem 2 are fulfilled. Let* (ϱ_j) *be any sequence of numbers* $\varrho_j \geqq \varrho_0$ *such that* $\lim_{j \to \infty} \varrho_j = +\infty$ *and let* u_{ϱ_j} *be a solution of* (1.12), (1.13) *for* $\varrho = \varrho_j$. *Then the sequence* (ϱ_j) *has a subsequence* (ϱ_j^*) *such that for any compact* $K \subset \mathbf{R}^n$

$$(2.1) \qquad \lim_{j \to \infty} \|u_{\varrho_j^*} - u^*\|_{H^{2m}(K)} = 0$$

holds where $u^* \in H_{\mathrm{loc}}^{2m}(\mathbf{R}^n)$ *is a solution of* (0.9) *vanishing at infinity.*

If the solution u of equation (0.9) *is unique then for the solutions* u_ϱ *of* (1.12), (1.13)

(2.2) $$\lim_{\varrho\to\infty} \|u_\varrho - u\|_{H^{2m}(K)} = 0$$

holds with arbitrary compact $K\subset\mathbf{R}^n$.

If estimations (0.2), (0.8) *are valid, too, then*

(2.3) $$\lim_{j\to\infty} \|u_{\varrho_j^*} - u^*\|_{H^{2m}(B_{\varrho_j^*})}$$

resp. (in the case of unicity)

(2.4) $$\lim_{\varrho\to\infty} \|u_\varrho - u\|_{H^{2m}(B_\varrho)} = 0$$

hold.

Proof. The solutions $v_\varrho\in L_a^2(\mathbf{R}^n)$ of the equation (1.14) constitute a bounded set in the Hilbert space $L_a^2(\mathbf{R}^n)$. If it were not true then there would exist a sequence $(v_{\varrho_j'})$ of solutions of (1.14) such that

(2.5) $$\lim_{j\to\infty} \|v_{\varrho_j'}\|_{L_a^2(\mathbf{R}^n)} = +\infty.$$

From (1.14) it is clear that

(2.6) $$\frac{v_{\varrho_j'}}{\|v_{\varrho_j'}\|_{L_a^2(\mathbf{R}^n)}} + \frac{G_{\varrho_j'}(v_{\varrho_j'})}{\|v_{\varrho_j'}\|_{L_a^2(\mathbf{R}^n)}} = \frac{f}{\|v_{\varrho_j'}\|_{L_a^2(\mathbf{R}^n)}}.$$

By (2.5) and (1.15) the term on the right and the second term on the left in (2.6) tend to the zero of $L_a^2(\mathbf{R}^n)$ as $j\to\infty$. The norm of the first term on the left equals one, thus from (2.6) we have a contradiction.

From the boundedness of the solutions v_ϱ of (1.14) and from (1.16) it follows the boundedness of the functions $u_\varrho = A_\varrho^{-1} v_\varrho$ in $H^{2m}(B_a)$.

Consider any sequence of numbers $\varrho_j \geqq \varrho_0$ such that $\lim\limits_{j\to\infty} \varrho_j = +\infty$. The sequence (u_{ϱ_j}) of solutions of (1.12), (1.13) with $\varrho=\varrho_j$ is bounded in the norm of $H^{2m}(B_a)$. Hence (u_{ϱ_j}) has a subsequence $(u_{\varrho_j^*}) = (u_j')$ which tends to a function $u_0\in H^{2m-1}(B_a)$ in the norm of $H^{2m-1}(B_a)$:

(2.7) $$\lim_{j\to\infty} \|u_j' - u_0\|_{H^{2m-1}(B_a)} = 0.$$

In view of (1.3) and the mean value theorem it is clear that

$$|g(x, u_j', \dots, D^\beta u_j', \dots) - g(x, u_0, \dots, D^\beta u_0, \dots)| \leqq$$
$$\leqq c_1 \sum_{|\beta|\leqq 2m-1} |D^\beta u_j' - D^\beta u_0|$$

(c_1 denotes a constant). Thus

(2.8) $$\lim_{j\to\infty} \int_{B_a} |g(x, u_j', \dots, D^\beta u_j', \dots) - g(x, u_0, \dots, D^\beta u_0, \dots)|^2\, dx = 0.$$

Consider the functions $v'_j = Au'_j$. Then

$$v'_j + g(x, u'_j, \ldots, D^\beta u'_j, \ldots) = f \tag{2.9}$$

since the functions u'_j are solutions of the problem (1.12), (1.13) for $\varrho = \varrho_j^*$. Equalities (2.8), (2.9) imply that the sequence (v'_j) tends to a function $v^* \in L_a^2(\mathbf{R}^n)$ in the norm of $L_a^2(\mathbf{R}^n)$ and

$$v^* + g(x, u_0, \ldots, D^\beta u_0, \ldots) = f. \tag{2.10}$$

We shall prove that for any compact $K \subset \mathbf{R}^n$

$$\lim_{j \to \infty} \|u'_j - A^{-1} v^*\|_{H^{2m}(K)} = 0. \tag{2.11}$$

Since $u'_j = A_{\varrho_j^*}^{-1} v'_j$, thus

(2.12)
$$\|u'_j - A^{-1} v^*\|_{H^{2m}(K)} \leqq \|A_{\varrho_j^*}^{-1} v'_j - A^{-1} v'_j\|_{H^{2m}(K)} + \|A^{-1}(v'_j - v^*)\|_{H^{2m}(K)}.$$

The sequence (v'_j) is bounded in $L_a^2(\mathbf{R}^n)$ hence by (0.7) the first term on the right in (2.12) tends to zero as $j \to \infty$. Applying the estimation (0.6) to $A^{-1}(v'_j - v^*)$ we find that the second term on the right in (2.12) tends to zero, too. Thus (2.12) implies (2.11).

From (2.11), (2.7) it follows that

$$u_0 = A^{-1} v^* \quad \text{a. e. in} \quad B_a. \tag{2.13}$$

Denote $A^{-1}v^*$ by u^* then $u^* = u_0$ a.e. in B_a, $v^* = Au^*$ and by use of (2.10) we find that

$Au^* + g(x, u^*, \ldots, D^\beta u^*, \ldots) = f$, further u^* tends to zero at infinity. Equality (2.11) implies the estimation (2.1).

Equality (2.2) can be proved as follows. Assume that the solution u of (0.9) is unique but equality (2.2) is not valid. Then there exist a compact $K \subset \mathbf{R}^n$, a number $\varepsilon_0 > 0$ and a sequence $(u_{\tilde{\varrho}_j}) = (\tilde{u}_j)$ such that $\lim_{j \to \infty} \tilde{\varrho}_j = +\infty$ and

$$\|\tilde{u}_j - u\|_{H^{2m}(K)} \geqq \varepsilon_0, \quad j = 1, 2, \ldots. \tag{2.14}$$

Then by use of the first part of the proof we have that $(\tilde{u}_j)$ has a subsequence $(\tilde{u}'_j)$ such that

$$\lim_{j \to \infty} \|\tilde{u}'_j - \tilde{u}\|_{H^{2m}(K)} = 0 \tag{2.15}$$

where $\tilde{u}$ is a solution of (0.9), vanishing at infinity. Since the solution of (0.9) is unique, thus $\tilde{u} = u$ and (2.14) is impossible because of (2.15).

If the estimations (0.2), (0.8) are valid, too, then it is easily seen that

$$\lim_{j \to \infty} \|u'_j - A^{-1} v^*\|_{H^{2m}(B_{\varrho_j^*})} = 0$$

(see the proof of (2.11)). This equality implies (2.3). (2.4) can be proved similarly if the solution of (0.9) is unique.

Remark. In [1] and [2] there are formulated sufficient conditions on P and Q which guarantee that the conditions in Theorem 2 and in Theorem 3 are fulfilled (see the introduction).

References

[1] Л. Шимон, Об аппроксимации решений граничных задач в неограниченных областях, *Дифференциальные уравнения,* **IX**, 8 (1973), 1482—1492.

[2] Б. Р. Вайнберг, Принципы излучения, предельного поглощения и предельной амплитуды в общей теории уравнений с частными производными, *Успехи Матем. Наук,* **23**, 3 (129) (1966), 115—194.

[3] L. Simon, On elliptic differential equations in $\mathbf{R}^n$, *Ann. Univ. Sci. Budapest,* **27** (1984), to appear.

[4] D. E. Edmunds and J. R. L. Webb, Quasilinear elliptic problems in unbounded domains, *Proc. Roy. Soc. London, Ser. A,* **337** (1973), 397—410.

[5] J. R. L. Webb, Boundary value problems for strongly nonlinear elliptic equations, *J. London Math. Soc.* (2), **21** (1980), 123—132.

[6] F. E. Browder, Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains, *Proc. Nat. Acad. Sci. USA,* **74** (1977), 2659—2661.

[7] F. H. Michael, An application of the method of monotone operators to nonlinear elliptic boundary value problems in unbounded domains, *Ann. Univ. Sci. Budapest,* **25** (1982), 69—84.

[8] F. H. Michael, An elliptic boundary value problem for nonlinear equations in unbounded domains, *Ann. Univ. Sci. Budapest,* **26** (1983), 125—139.

DEPARTMENT OF ANALYSIS II
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY

Bibliographie

Rainer E. Burkard—Ulrich Derigs, Assignment and Matching Problems: Solutions Methods with FORTRAN Programs, (Lecture Notes in Economics and Mathematical Systems, 184) VIII+48 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

Assignment and matching problems belong to those combinatorial optimization problems which are well understood in theory and have many applications in practice. This book deals with some well-known assignment and matching problems studied by a research group of the Mathematical Institute in Cologne.

The problems studied are as follows: the Linear Sum Assignment Problem, the Linear Bottleneck Assignment Problem, the Cardinality Matching Problem, the Sum Matching Problem, the Bottleneck Matching Problem, the Chinese Postman Problem, Quadratic Assignment Problems.

All of the above problems have been solved by exact methods, exept the last one which was solved by heuristic methods of two different types. After presenting the theoretical background of the methods for these problems, the authors give the input and output lists of the FORTRAN-programs which were extensively tested on a CDC CYBER 76 in Cologne and on an IBM 4331 of the Sonderforschungbereich 21 at the University of Bonn. The book contains detailed references and the descriptions of FORTRAN-programs of all of the presented problems.

The programs are correct but there are some regrettable misprints (for example on page 36, on Figure 3.1 the edges (21, 23) and (15, 17) are missing).

Appart from these unimportant misprints the book is very useful for readers who can save a lot of time during the implementation of these algorithms.

G. Galambos (Szeged)

Complex Analysis, Methods, Trends and Applications, Edited by E. Lanckau and W. Tutschke, 398 pages, Akademie-Verlag, Berlin, 1983.

Holomorphy is the basic concept of today's complex analysis. The enlarged effectiveness of the concept of holomorphy has produced new results, methods and applications. For example, in the case of nonlinear elliptic differential equations there are 1—1 mappings between the solutions and holomorphic functions. It is possible to construct solutions and to describe the properties of given solutions with the help of solutions of corresponding problems for holomorphic functions. Thus, a nonlinear problem is reducible to a linear one.

This book presents new methods of complex analysis, and it compares and connects these methods with classical ones. The main tendencies, which express new relations between complex analysis and the theory of partial differential equations, are described in the book. It is written by an international team (24 authors). In the first theoretical part 8 chapters deal with value distribution theory, polyanalitic functions, cohomological methods, approximation methods and other problems of complex analysis. The 14 chapters of the second part contain various applications, especially to partial differential equations.

The book is directed not only to specialists in complex analysis but also to all mathematicians, physicists and to all scientists, who are interested in: analysis, in general.

T. Krisztin (Szeged)

K. J. Devlin, Fundamentals of Contemporary Set Theory, VIII+182 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1979.

This book provides an account of those parts of set theory which are of direct relevance to other areas of pure mathematics. The main emphasire is put on the question of how to axiomatize set theory. Therefore, parts of naive set theory are developed first (Chapter I). This is followed by the definition of a language of set theory (LAST), whereafter the axioms are presented (with motivations) in detail. Classes are described as certain formulas of the language LAST. Thereafter the Recursion Principle is discussed. Intiutively, it says that, in ZF, it is possible to define functions with recursion. A restricted version of this principle is proved first as a theorem scheme in LAST. Then the Recursion Principle is presented in full generality with a sketch of the proof. Chapter II is completed with a discussion of the Axiom of Choice. Chapter III develops the theory of ordinal and cardinal numbers. Chapter IV deals with some topics of pure set theory such as stationary sets, regressive functions, trees, etc. Chapter V discusses the Axiom of Constructibility, its consistency as well as the consistency of AC and of GCH. Proofs are omitted, but indications, how some proofs proceed, are given. Chapter VI sketches the proof of the independence of GCH by exhibiting a Boolean valued model.

This is a very useful book on the fundamentals of set theory containing a number of helpful comments which are not available elsewhere.

A. P. Huhn (Szeged)

Differential Equations, Proceedings, Sao Paulo, 1981, Edited by D. G. de Figueiredo and C. S. Hönig (Lecture Notes in Mathematics, 957), VIII+301 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

This volume contains 14 papers presented at the 1st Latin American School on Differential Equations, held at Sao Paulo, Brazil, June 29—July 17, 1981. The four longer papers comprise the lectures of the courses, which were delivered by A. Castro B. (Reduction methods via minimax), D. G. de Figueiredo (Positive solutions of semilinear elliptic problems), J. Ize (Introduction to bifurcation theory) and P. H. Rabinowitz (The mountain pass theorem: theme and variations). The authors of the other 10 papers are A. Castro B. and J. V. A. Goncalves, S. Hahn-Goldberg, D. B. Henry, C. S. Hönig, A. F. Ize, J. Lewowicz, P. S. Milojevic, G. P. Menzala. L. L. Schumaker, J. Sotomayor. These papers are original research papers in different fields of differential equations and in the general field of mathematical analysis.

T. Krisztin (Szeged)

R. E. Edwards, Fourier Series. A Modern Introduction, Vol. 2 (Graduate Texts in Mathematics, 85), xi+369 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

This is the second edition of a book appeared first in 1967. There are numerous minor corrections and changes. In addition, a substantial reformulation and up-dating of Chapters 14 and 15 has taken place.

The volume deals on the whole with the more modern aspects of Fourier series and related topics that fit most naturally in a functional analytic context. With their introduction to distributional concepts and techniques and to interpolation theorems, Chapters 12 and 13 are perhaps the most significant portions of Volume 2. The carefully detailed discussion of the Marcinkiewicz interpolation theorem renders this topic more accessible to a beginner. The major part of Chapter 11 is devoted to the elements of Banach algebra theory and its applications in harmonic analysis. In the final Chapter 16 there appears an unusually well-connected introductory account of multiplier problems and related matters. Chapter 14 is concerned with random Fourier series. A greater coherency is attained by involving harmonic analysis on the Cantor group. Chapter 15 is devoted to the study of lacunary Fourier series. Much of these two chapters is independent of the previous ones, or is easily made so.

Each chapter ends with a large number of exercises covering a wide range of difficulty. The more difficult ones are provided with hints to their solutions. The thorough bibliography comprising 25 pages contains many suggestions for further reading. There is a cross-referencing system in the two volumes. The treatment is supplemented by a list of Symbols and a long Index.

To sum up, with its companion volume, Fourier Series II provides a vital exposition of various aspects of harmonic analysis over the circle group which are of current research interest. It can be well adapted to course work, too. We highly recommend both volumes to graduate students who wish to continue studies as well as to research workers in Fourier Analysis.

F. Móricz (Szeged)

P. J. Federico, Descartes on Polyhedra, A Study of the "De Solidorum Elementis", Sources in the History of Mathematics and Physical Sciences 4, x+145 pages, with 36 figures, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

This book is based on the "De solidorum Elementis", a short but extremely interesting work of Descartes, which has survived in a manuscript of Leibniz only. In this work Descartes developed some new ideas in the arithmetization of geometry. It is not the first edition of this text, but it differs from its predecessors. The present work is based on a fresh examination of the manuscript. The complete facsimile of the manuscript and the first English translation of it are contained in this volume as well as commentaries of the text. In a section the author compares the works of Euler and Descartes on polyhedra.

This book and the other volumes in this series are warmly recomended to everybody who is interested in the history of mathematics, especially in original texts.

Lajos Klukovits (Szeged)

Jens E. Fenstad, General Recursion Theory. An Axiomatic Approach (Perspectives in mathematical logic) Springer-Verlag, Berlin—Heidelberg—New York, 1980.

General recursion theory, in particular, the axiomatic development of computational theories is of primary interest in several different areas of logic and computer science. The author aims to present such a development. Quoting his words: "The book is introductory. The aim is to provide a reasonably unified view. Not the only possible one, but one broad and detailed enough to serve as a basis and general framework."

Central to the discussions is an analysis of the relation $\{a\}(\sigma)\simeq z$, which is to be understood as the "computing device" coded by "a gives output z for the input σ". In a preliminary chapter.

entitled Pons Asinorum, the author gives an extremely clear exposition on his ideas concerning the very nature of the relation above. In fact, he presents not only his concise views on the matter at an informal level, but he also relates them to and locates them among other approaches, such as inductive definability, computation theories and Platek's theory on computations in higher types.

The matterial is devided into four parts (disregarding the expository part), each consisting of two chapters. Chapter 1 deals with the combinatorial part of the general theory. Subcomputations and length of a computation enter the picture in Chapter 2. These first two chapters cover the fundamental representation theorems and a general version of the first recursion theorem. The next part is devoted to finite theories on one (Chapter 3) and two (Chapter 4) types. The first of them corresponds to a generalization of hyperarithmetic theory while the second one is a general version of recursion in higher types, or second order definability.

Part C treats infinite theories, that is, generalizations of ordinary recursion theories and those of admissibility theories, including general degree theory and some recent results in inadmissibility theory. The last part of the volume discusses set recursion and computations in higher types.

The book is excelently written, most of the material is new, and in the reviewer's belief, it would be enjoyable to read it to everyone, including experts and students, interested in recent developments and ideas of recursion theory.

P. Ecsedi-Tóth (Szeged)

Jörg Flum—Martin Ziegler *Topological Model Theory.* (Lecture Notes in Mathematics 769), Springer-Verlag, Berlin—Heidelberg—New York, 1980. p. X+151.

An algebraic structure enriched with a topology on its universe is called a topological structure. To study topological structures, a formal language L_t is introduced as a fragment of the monadic second order language, in which set variables range over the topology and second order quantification is allowed over small neighborhoods of a point.

All interesting topological concepts are expressible in L_t, while, on the other hand, L_t is mathematically tractable as opposed to other more general second order languages. In fact, L_t is the largest extension of the first order language with set variables appropriate to investigate topological structures and yet to possess compactness and Löwenheim-Skolem properties. Several methods and results known from the model theory of first order languages apply directly to L_t. The first part of this volume is devoted to import these results and methods to L_t.

The authors start with overwieving preliminaires, and in particular, with introducing the concept of invariant second order formulae which play a central role in their considerations. It turns out, that several topological notions are invariant. Moreover, it is proved in §§2, 4, that any formula of L_t is invariant and conversely, any invariant formula (of the full monadic second order language) is equivalent to a formula of L_t. Section 3 is devoted to translating the language L_t into a two-sorted first order language in a truth-value preserving way thus establishing some basic results of L_t like compactness and Löwenheim-Skolem theorems. In the next section, two algebraic characterization of L_t-elementary equivalence are given. Sections 5, 6 deal with interpolation and preservation theorems under dense and open substructures, products and sums of topologies. The next section is devoted to definability problems concerning both classical explicit definability in topological structures and explicit definability of topologies. In Section 8, several other topological languages are compared and a Lindström-type characterization is given for L_t. The omitting types theorem is proved for a fragment of L_t in Section 9. It is also shown, that this theorem fails to hold in L_t. Finally, the infinitary version $(L_{\omega_1\omega})_t$ of L_t is investigated in the last section of the first part and several results obtained in earlier sections are generalized to this case.

The second part of the volume is devoted to applications of the first one to topological spaces and in particular to topological abelian groups, fields and vector spaces. In the last section, topological vector spaces are investigated. Firstly, an L_t axiomatization of locally bounded real topological vector spaces is given and is proved to be complete if the dimension is fixed. Secondly, it is shown, that the L_t-theory of surjective and continuous linear mappings is axiomatizable.

The book seems to be a basic reference for researchers in several different areas of model theory, algebra and topology. The matterial is clearly presented (disregarding some misprints), concise and easily comprehensible, hence the text can be useful for experts as well as graduate students.

P. Ecsedi-Tóth (Szeged)

Geometry and Analysis. Papers Dedicated to the Memory of V. K. Patodi, V+166 pages, Published for the Indian Academy of Sciences, Bangalore and Tata Institute of Fundamental Research, Bombay, Springer-Verlag, New York—Heidelberg—Berlin, 1981.

The early death of the brilliant differential geometer Vijay Kumar Patodi (1945—1976) is a great loss not only to his mother country India but also to the international mathematical community. The book is a collection of papers, intended to be a mathematical tribute to the memory of the great mathematician. It contains articles by V. Arnold (On some problems in singularity theory), M. F. Atiyah and R. Bott (Yang-Mills and bundles over algebraic curves), J. Dodziuk (Vanishing theorems for square-integrable harmonic forms), J. J. Duistermaat (On operators of trace class in $L^2(X, \mu)$), J. Eells and L. Lemaire (Deformations of metrics and associated harmonic maps), Peter B. Gilkey (Curvature and the heat equation for the de Rham complex), H. P. McKean (Units of Hill curves), Harish-Chandra (A submersion principle and its applications), J. J. Millson and M. S. Raghunathan (Geometric construction of cohomology for arithmetic groups I), M. S. Narasimhan and M. V. Nori (Polarisations on an abelian variety), S. Raghavan and S. S. Rangachari (Poisson formulae of Hecke type), S. Ramanan (Orthogonal and spin bundles over hyperelliptic curves).

The book also contains the biography and the list of publications of V. K. Patodi and his photograph.

Z. I. Szabó (Szeged)

T. W. Hungerford, Algebra (Graduate Texts in Mathematics, 73), XXIII+502 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1980.

In the preface of this volume the author gives exactly the motivations of writing the book. (This is the second edition of the first *Algebra,* published in 1974, Holt, Rinehart and Winston, Inc. No substantial changes have been made, only misprints and errors have been corrected and some proofs have been rewritten.) He feels that a student learning algebra at the beginning graduate level or anyone who wants to get acquainted with beginning abstract algebra needs a textbook containing the basic material in sufficient breadth and depth, the knowledge of which is essential for studying special topics of algebra or the applications of abstract algebra in other mathematical sciences. Years ago, the author was unable to find a suitable textbook for such aims. So he decided to write one, trying to fill this long-felt gap. In our opinion he succeeded very well in carrying out this plan, not only in a masterful matching but in giving such a treatment, which makes easy to adapt the material to the modern universal algebraic and category theoretic framework. A very large number of exercises is offered at the end of every section; the full treatment of these surely requires skill and a steady knowledge of the corresponding topic. But some results of classical higher algebra are also contained here (e.g. the equations of degree $n \leqq 4$, Lagrange's interpolation formula and so on), so the solution of at least these exercises is important.

The book consists of an Introduction and 10 chapters. The Introduction contains the necessary set-theoretical concepts and statements, such as sets, maps, set operations, Zorn's lemma, and elementary cardinal number arithmetic. The chapters are I. Groups, II. The structure of groups, III. Rings, IV. Modules, V. Fields and Galois Theory, VI. The Structure of Fields, VII. Linear algebra, VIII. Commutative Rings and Modules, IX. The Structure of Rings, X. Categories; the interdependence of these is given by the author. For those interested only in the most frequent types of structures and the possibility of the simplest applications of abstract algebra it is sufficient to read the Introduction and chapters I, III, IV and VII. The further chapters are dealing in more details with groups, rings, fields and modules, and treat special kinds and problems of these (fundamental theorem of finitely generated Abelian groups, the Krull—Schmidt Theorem, Sylow's theorems, field extensions, Galois groups, the general equations of degree n, transcendence degree, separability, Noetherian rings and Dedekind domains, Hilbert's famous theorem on proper ideals of a polynomial ring (Hilbert's Nullstellensatz), prime and Jacobson radicals of rings, simple and semisimple rings, structure and characterization theorems for left Artinian and Noetherian semisimple rings, and in Chapter X, morphisms, functors, adjoint functors). We call particular attention to Chapter VII, which is an excellent and brief summary, from an abstract point of view, of the most important results of a traditional area, the theory of matrices, linear transformations and determinants.

The book is recommended for university students and for those interested in abstract algebra, and for research workers, too. Reading of this volume requires no preliminary knowledge from higher mathematics but a high mathematical intelligence, ability of doing abstract considerations. The full attainment of the book gives a very good base for studying e.g. universal algebra (with an additional acquaintance in lattice theory) and other modern branches of algebra. A bibliography partitioned according to different topics gives good instructions for further studies.

Attila Lenkehegyi (Szeged)

Iterative Solution of Nonlinear Systems of Equations (Proceedings, Oberwolfach 1982), Edited by R. Ansorge, Th. Meis, and W. Törnig (Lecture Notes in Mathematics, 953), VII+202 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

The conference indicated in the title was held in the "Mathematisches Forschungsinstitut Oberwolfach" (Federal Republic of Germany) between January 31 and February 5, 1982. In all twenty four talks were given, thirteen of which are presented in these proceedings.

The table of contents: 1. O. Axelsson: On global convergence for nonlinear problems. 2. W. Hackbusch: Multi-grid solution of continuation problems. 3. H. D. Mittelmann: A fast solver for nonlinear eigenvalue problems. 4. H. Cornelius and G. Alefeld: A device for the acceleration of convergence of a monotonously enclosing iteration method. 5. B. Kaspar: Overrelaxation in monotonically convergent iteration methods. 6. A. Neumaier: Simple bounds for zeros of systems of equations. 7. K. Nickel: Das auflösungsverhalten von nichtlinearen Fixmengen-Systemen. 8. F. A. Potra: On the convergence of a class of Newton-like methods. 9. U. Hornung: ADI-methods for nonlinear variational inequalities of evolution. 10. G. Kolb and W. Niethammer: Relaxation methods for the computation of the spectral norm. 11. Th. Meis and W. Baaske: Numerical computation of periodic solutions of a nonlinear wave equation. 12. C. Weiland: Erfahrungen bei der Anwendung numerischer Verfahren zur Lösung nichtlinearer hyperbolischer Differentialgleichungssysteme. 13. W. Werner: On the simultaneous determination of polynomial roots.

Emphasis lies on three main topics: (1) multigrid methods (talks 1—3), (ii) monotone and interval arithmetic iterations (talks 4—8), (iii) applications in industrial practice (talks 9—12). The book provides an up-to-date account of the present stage of the subject. We warmly recommend it to everybody, who works in Numerical Analysis and/or in Applied Mathematics in Engineering.

F. Móricz (Szeged)

J. H. van Lint, Introduction to Coding Theory (Graduate Texts in Mathematics, Vol. 86), IX+171 pages with 8 illustrations, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

A gap has been filled: here is a book on conding theory written for outsiders. Many mathematicians and engineers have to study coding theory since it belongs to the most up-to-date part of applied mathematics. A number of authors referred to in the book are (former) members of the staff of Bell Telephone Laboratories. It is explained how satellites can transmit excellent pictures which were taken of Mars and other planets to Earth.

This book is a revised and enlarged edition of van Lint's previous book Inleiding in de Coderingstheorie.

A fairly thorough mathematical background is required to read this book. In the first place this means algebra, but elements of combinatorics and probability theory are required as well.

The existence of good codes is proved by random techniques and many codes are constructed, e.g. Hadamard codes, Reed—Muller codes, Golay codes, BCH codes, Reed—Solomon codes, quadratic residue codes, Goppa codes and the major development in coding theory in the seventies: Justesen codes.

Several bounds are given for the codes. The book is provided with exercises, hints and complete solutions are contained. It is of interest to mathematicians and computer scientists, students of mathematics and computer science.

L. A. Székely (Szeged)

Uwe Kastens—Brigitte Hutt—Erich Zimmermann, GAG: A Practical Compiler Generator, (Lecture Notes in Computer Science, 141), Springer-Verlag, Berlin—Heidelberg—New York, 1982.

Attribute grammars have proved useful tools for specifying programming languages and their implementations in compiler writing systems. The GAG-System is a compiler *G*enerator based on *A*ttribute *G*rammars. In general, the main problem connected with attribute grammars is to produce an efficient attribute evaluator. During the parsing a parse tree is constructed and the evaluator computes the values of the attribute instances attached to the parse tree. In the GAG-System the OAG (Ordered Attribute Grammars) attribute evaluation technique was implemented. Using this technique efficient evaluators can be generated for a large subclass of the non-circular attribute grammars.

The space management technique of GAG finds those attributes which can be implemented by global variables or stacks. Thus, the space requirements of the generated compilers are close to those of conventional compilers.

The metalanguage of the GAG-System is based on a powerful type concept and the system was implemented in Pascal. The complete description of Pascal in GAG can be seen in Appendix *A*.

Tibor Gyimóthy (Szeged)

Tosio Kato, A Short Introduction to Perturbation Theory for Linear Operators, XI+161 pages, Springer-Verlag (New York—Heidelberg—Berlin), 1982.

Professor Kato's excellent *Perturbation Theory for Linear Operators* (for a review cf. *these Acta*, **40** (1978), p. 398) is a Bible of this branch of Operator Theory and, accordingly, it is rather voluminous. Readers who are interested in a shorter introduction to the subject, not going beyond the case of finite-dimensional spaces, will find it convenient to have now this handy shorter version. Actually, it contains the first two chapters of the original. As the author states in the Preface, these two chapters were intended from the outset to be a comprehensive presentation of those parts of perturbation theory that can be treated without the topological complications of infinite-dimensional spaces. What results is an interesting introduction to linear algebra, which systematically uses complex functions, by way of the resolvent theory. Of course, not all parts of Perturbation Theory have non-trivial contents in finite-dimensional spaces. Such are, in particular, the parts pertaining to scattering problems.

Some new sections and paragraphs have been added, e.g. on product formulas, dissipative operators and contraction semigroups, positive matrices, etc.

The booklet is not, and cannot be, a condensation of the whole theory, but, within its restricted framework, it is appealing to a wider audience and will certainly be a welcome addition to the existing literature.

Béla Sz.-Nagy (Szeged)

Jesper Lützen, The Prehistory of the Theory of Distributions (Studies in the History of Mathematics and Physical Sciences 7), VIII+232 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

This book is not primarily a story about how Sobolev and Schwartz developed the theory of distributions. All the important techniques and theories (generalized derivatives and generalized solutions to differential equations, generalized Fourier transforms, early generalized functions, de Rham's currents), which anticipated the theory of distiibutions, are discussed. These four theories and their connection to the theory of distributicns constitute the bulk of the book. The author says: "Did Sobolev and Schwartz construct distributions from scratch or were there earlier trends and, if so, what were they? It is this, concerning the prehistory of the theory of distributions, which I attempt to answer in this book".

The book is divided into six chapters. In the first the development of functional analysis is summarized. The next four chapters discuss the four main trends of the prehistory mentioned above. The last chapter deals with L. Schwartz's creation of theory of distribution. Each of the six chapters can be read separately. Thus, this book may well be of interest to readers who are only interested in the work of Sobolev and Schwartz or in one of the four trends of the prehistory.

T. Krisztin (Szeged)

Yu. I. Manin, A Course in Mathematical Logic (Graduate texts in mathematics; 53), Springer-Verlag, New York—Heidelberg—Berlin, 1977.

This book, translated from the Russian original by N. Koblitz, leads the reader from the very beginning of mathematical logic up to recent discoveries, including the independence of the continuum hypothesis and the result on the Diophantine nature of enumerable sets and others.

The first two chapters are a beginner's course in predicate logic with outstanding clarity, and several useful explanations on the background ideas. Central to the presentation are the concepts of models, truth and some other semantical notions. In the last part of Chapter 2, using a method due to Smullyan, the author proves Tarski's theorem on the undefinability of truth in arithmetic without introducing recursive functions. This theorem will be the main tool for proving Gödel's incompleteness result in a later chapter. Also, Chapter 2 contains a section dealing with quantum-logic. The third and fourth chapters are devoted to the complete and detailed proof of the independence of the continuum hypothesis by developing constructible sets of Gödel in the von Neumann's cumulative hierarchy and Boolean-valued models due to D. Scott for presenting Cohen's forcing. Recursive functions, enumerable sets, Church's thesis and some problems of algorithmic undecidability are treated in Chapter 5. The next chapter continues the study of enumerable sets by establishing a recent result on their Diophantine nature, and in particular, the existence of undecidable enumerable sets. In the last section, the very interesting (and in logical textbooks mostly neglected) theory concerning the length of proofs is presented following the ideas of Gödel and Kolmogorov. The seventh chapter is devoted to Gödel's incompleteness theorem and contains several detailed explanations on the significance of this result. Finally, in the last chapter of the volume, recursive structures, in particular, recursive groups are discussed in an attractive way.

This book seems to be an excelent introduction to modern applications of mathematical logic for graduate students.

P. Ecsedi-Tóth (Szeged)

Mathematical Modeling of the Hearing Process. Proceedings, Troy, NY 1980, Edited by Mark H. Holmes and Lester A. Rubenfeld (Lecture Notes in Biomathematics, 43), V+104 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

The cochlea, making possible hearing to humans and other mammals, is a mystic instrument which nevertheless is phisical enough to allow various deterministic mathematical models. Cochlear mechanics exists since Helmholtz, but his models were replaced by the "long wave" theories in the 1950's. These models came into question since the early 1970's following Rhode's measurements of the vibration of the basilar membrane in monkeys. According to the classification of the editors, cochlear mechanics is presently in the third stage of its history. The six papers in these proceedings provide a comprehensive overview of this third stage of research in the mathematical modeling of the hearing process. Beside experts in hearing, the volume may be of interest to workers in vibration mechanics and in differential equations.

Sándor Csörgő (Szeged)

Numerical Integration of Differential Equations and Large Linear Systems (Proceedings, Bielefeld 1980), Edited by Juergen Hinze (Lecture Notes in Mathematics, 968), VI+412 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

The use of electronic computers has formed a firm basis for extensive mathematical experiments in the field of numerical solution of differential equations. The resulted developments in the methods are carried out not only by numerical analysts in general, but also by chemists, physicists and engineers. The latter, faced with specific problems, have developed selective methods, which appear to be the most suitable for the specific problem at hand.

It was the purpose of the two consecutive workshops on "Numerical Integration of Differential Equations" and "Large Linear Systems; Eigenvalue and Linear Equations" held at the

"Zentrum für interdisziplinäre Forschung" of the University of Bielefeld (Federal Republic of Germany) in spring 1980, to bring together numerical analysts and chemical physicists in order to make a further progress of the numerical methods used in chemical physics. The same purpose is to be surved by this volume, a proceedings of these workshops. It contains twenty eight papers. To emphasize the interdisciplinary character, the first ten papers included focus on specific applied problems in chemical physics. Besides, valuable additional information on the numerical methods used in scattering theory can be found in the first four contributions. The following ten papers are devoted to specific improvements in the methodology of integrating various types of differential equations and error estimates of bounds for such methods. The major emphasis in the procedures is on finite difference methods. Since every discretization algorithm of differential equations leads to a large linear equation or eigenvalue problem, the main theme for the last eight papers is the efficient solution of such large linear systems, where the coefficient matrices are of special stucture or sparse.

To sum up, the present volume deals with numerical methods useful in atomic and molecular physics, in particular scattering calculations and the solution of the coupled differential equation in chemical kinetics. Thus, it is highly recommended to all who work in one of these fields, and without doubt, it must be found in every applied mathematics library.

F. Móricz (Szeged)

Probability in Banach Spaces IV. Proceedings, Oberwolfach, Germany. Edited by A. Beck and K. Jacobs (Lecture Notes in Mathematics, 990), V+234 pages, Springer-Verlag, Berlin—Heidelberg —New York—Tokyo, 1983.

This volume contains 16 papers read at the seminar on probability in Banach spaces held in Oberwolfach, July, 1983. This meeting was the fourth in a series of meetings on the above subject (Oberwolfach 1975, 1978 and Medford, Massachusetts, 1980) which fact well illustrates that geometric aspects of probability became a very rapidly growing field of recent research. This Oberwolfach Conference probed the conjection of probability and geometry, testing the contribution of geometrical assumptions to the classical type results of probability. The papers expose various directions of the subject and they present the latest results in the field. The technically prepared non-expert may use Woyczynski's survey on the asymptotic behaviour of sums of independent random vectors and Dettweiler's longer expository papers on Banach space valued processes with independent increments as a guide to join to the research on this area.

Lajos Horváth (Szeged)

Probability Theory and Mathematical Statistics. Proceedings, Tbilisi, USSR, 1982. Edited by K. Ito and Yu. V. Prokhorov (Lecture Notes in Mathematics, 1021), VIII+747 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983.

The Fourth USSR—Japan Symposium on Probability Theory and Mathematical Statistics was held in the Georgian SSR, August 23—29, 1982. This volume contains a part of the papers presented at the conference. Altogether 73 short papers are published in the volume, so the topics treated virtually cover present day Japanese and Soviet probability and statistics. For information we only list the section headings: Limit theorems, Stochastic equations and martingales, Mathematical statistics, Statistical physics, Stochastic analysis, Stability of stochastic models, Statistics of random processes, Stochastic control, Queueing theory, Ergodic theory, Branching processes, Gaussian processes, Semi-Markov processes, Random fields, Probability distributions and Noncommutative probability.

Lajos Horváth (Szeged)

Séminaire de Probabilités XVII, 1981/82. Proceedings. Edité par J. Azéma et M. Yor (Lecture Notes in Mathematics 986), V+512 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983.

This volume continues the traditional Strasbourg—Paris seminar notes and, of course, the main subject is the traditional general theory of stochastic processes. The topics cover a broad part of recent probability theory (local time of stochastic processes, stochastic differential equations, stochastic calculus, martingales, stopping times, Markov processes and invariance principles), so everybody doing research in probability theory can find something interesting and useful for himself. The last page contains corrections to the previous volumes of the Séminaire de Probabilités.

Half of the papers presented in this book are written in French without English summary. I believe that most readers would find useful English abstracts attached to these papers.

Lajos Horváth (Szeged)

Colin Sparrow, The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors (Applied Mathematical Sciences, 41), XII+269 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

This book is concerned with the system of differential equations $\dot{x}=\sigma(y-x)$, $\dot{y}=rx-y-xz$, $\dot{z}=xy-bz$, where σ, r and b are three real positive parameters. One might think: "How is it possible to write a whole book on a particular differential system, even which seems quite simple? If it is not integrable by quadratures, or does not allow to get phase portrait by methods of the qualitative theory, then it has to be integrated numerically by computers." The system was obtained by E. N. Lorenz, who is a meteorologist as well as a mathematician, in 1963 in the course of modelling a two dimensional fluid cell warmed from below and cooled from above. During the numerical experiments something strange happened: for wide ranges of values of the parameters, approximate solutions to the equations calculated on a computer looked extremely complicated. Unfortunately, presenting figures is beyond the frame of this review (the reader can find 91 illustrations in the book!) so we have to illuminate the behaviour of the solutions in words. The trajectories in the phase space R^3 were not periodic. Nevertheless, however long the numerical integration had been continued, the trajectories continued to wind around and around, first on one side, then on the other, without ever settling down to either periodic or stationary behaviour. The strange feature of the phenomenon consists in the fact that the equations are deterministic, they contain no random, noisy or stochastic terms and yet the trajectories are chaotic. As the author writes: "The suggestion, that complicated 'turbulent' behaviour in systems with an infinite number of degrees of freedom (such as atmosphere) might be modelled by simple deterministic finite-dimensional systems is one of the reasons why the Lorenz equations have attracted so much attention." The book interprets many different kind of chaotic behaviour that have been observed by other authors, and gives a global, geometric and intuitive understanding of these phenomena.

These notes are accessible also to readers possessing only the most basic concepts of the theory od differential equations. They can be suggested for every user and mathematician interested in differential system-models, because they excellently illustrate an approach to study many other systems behaving in ways which seem to be very similar to one or more of the behaviours shown by the Lorenz equations.

L. Hatvani (Szeged)

Stability Problems for Stochastic Models, Proceedings of the 6th International Seminar Held in Moscow, USSR, April 1982. Edited by V. V. Kalashnikov and V. M. Zolotarev (Lecture Notes in Mathematics, 982), XVII+295 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo.

This is a carefully compiled volume containing 22 articles. The average mathematical level of the papers included is much higher than what is usual in similar proceedings, and this fact makes tolerable the sometimes very bad English. More then half of the papers deal with characterisations of univariate or multivariate probability distributions and the stability of these characterisations. There are papers on stability of limit theorems, robust estimation, queueing systems, estimation of the parameters of stable distributions, on the $D(0, \infty)$ space, and three papers deal with probability metrics in a manner initiated by Zolotarev. In his nice foreword, emerging to a research paper, Zolotarev provides a broad notion of the stability of stochastic models, into which most of the specific problems above fit conveniently, and illustrates his method of metric distances for a concrete stability problem involving extreme value distributions.

Sándor Csörgő (Szeged)

The Mathematical Gardner, Edited by David A. Klarner, VIII+382 pages, Wadsworth International, 1981.

This book contains articles dedicated to Martin Gardner for his 65th birthday.

Martin Gardner is one of the world's greatest popularizer of mathematics. For more then twenty years he had had a column in each issue of Scientific American entitled Mathematical Games. (Now this appears in alternate issues.) These columns present old and new problems in a popular way. The articles contain examples, puzzles and questions answered in the next issue. Gardner's inimitable style makes the articles interesting for specialists and amateurs stimulating their creative activity. Gardner's books reached great success too, they are translated into several languages.

The articles in this book — edited by D. A. Klarner — are written by well-known mathematicians and are "real Gardners". The six chapters are: Games, Geometry, Two-Dimensional Tiling, Three-Dimensional Tiling, Fun and Problems and, finally, Numbers and Coding Theory. Here are some of the articles (following their original order in the book) which were extraordinarily interesting for the referee. "A Kriegspiel Endgame" is written by J. Boyce. Kriegspiel is a variant of chess: each player tries to mate his opponent, but neither player knows where the other player's pieces are. V. Chvátal's article "Cheap, Middling or Dear" especially shows how games could motivate readers toward deeper mathematics. The paper of S. Burr entitled "Planting Trees" illustrates the charms of combinatorial geometry. W. T. Tutte wrote an article on dissections into equilateral triangles. B. Grünbaum and G. C. Shephard give an excellent survey on some problems on plane tilings. H. S. M. Coxeter's paper "Angels and Devils" on mathematics and on Escher's work is beautiful aesthetically as well. In the introduction of his article entitled "My life among the polyominoes" D. A. Klarner relates Martin Gardner's role in his mathematical development. Some mathematical gems can be found in R. Honsberger's article. A new proof of Chvátal's art gallery theorem given by. S. Fisk is especially interesting for the referee. Surely, the reader will find some of the other articles more interesting than the above mentioned ones.

In conclusion, this is a book that certainly should be in every library.

L. Pintér (Szeged)

J. Uhl—S. Drossopoulou—G. Porsch—G. Goos—M. Dausmann—G. Winterstein—W. Kirschgässner, An Attribute Grammar for the Semantic Analysis of Ada. LNCS, Vol. 139, 511 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

Attribute grammar is an established tool for the formal specification of the semantics — generally in the form of a compiler — of a programming language. The symbols of the context-free grammar are associated via this specification with the groups of "attributes" expressing the various properties of the corresponding constructs of the programming language. This book contains an attribute grammar specifying the static semantics of ADA, as published in July 1980. An informal review of the GAG (Generator for Attribute Grammars) system is as follows. The metalanguage used by the GAG system is called ALADIN (A Language for Attribute Definition). The Karlsruhe Ada Compiler front-end (generated by GAG) consists of:

— The scanner (lexical analysis), which analyzes the input text and recognizes all lexical tokens of Ada.
— The parser (syntactic analysis), which generates from the sequence of lexical units the equivalent Diana Parse Tree with all lexical attributes (for example, identifier codes). The parser contains a complete error recovery system.
— The semantic analyzer, which reports any statically detectable error in the Ada program and calculates the semantic attributes required by Diana. It starts with the program representation by a Diana Parse Tree and adds the semantic attributes to this tree. These attributes convey information about the meanings of several language elements to subsequent compiler phases or to other tools of an Ada programming environment.

The front-end has, therefore, two internal interfaces: the list of lexical units as the output of the scanner, and the *structure tree* as the output of the parser. The structure tree represents the abstract syntax with the lexical information, and it is called the *Diana Parse Tree.*

The output of semantic analysis is the structure tree with lexical *and* semantics attributes. This abstract data type is called the *Diana Tree.* Semantic analysis mainly deals with name analysis (overloading resolution), type-checking and context conditions.

The complete static semantics of Ada is given by using attribute grammar and the description is written in ALADIN language.

Endre Simon (Szeged)

M. I. Yadrenko, Spectral Theory of Random Fields. Translation Series in Mathematics and Engineering, III+259 pages, Optimization Software, Inc., Publications Division, New York, 1983. (Distributed by Springer-Verlag, Berlin—Heidelberg—New York—Tokyo.)

This is the first volume in this new translation series and, according to Series Editor A. V. Balakrishnan, "typifies aboundantly" the aim of "this new series: to combine the best in mathematics and engineering, emphasizing the theoretical while strongly oriented toward applications". The latter orientation is only potential in this book, though, no doubt, many people from the engineering side who possess the necessary prerequisites to read it may find it very useful. The book is indeed first class mathematics both in its depth and in the arrangement of the material. It is really the first one in its kind and virtually covering the present state of knowledge in this important and vigorously developing area, it fills a long apparent gap in the literature.

The unified spectral theory of the 104 pages Chapter I (with section headings: Homogeneous and isotropic random fields, Spherical Averages of homogeneous and isotropic random fields, Homogeneous and isotropic random fields of the Markov type, Homogeneous and isotropic random fields in Hilbert space, Isotropic random fields on spheres, Isotropic random fields on Euclidean spaces, Strong law of large numbers for isotropic random fields) will surely become indispensable for both students and experts in the field as a main source of reference. Following a short Chapter II on the local behaviour of sample functions of random fields, Chapter III deals with absolute continuity and singularity of measures corresponding to random fields. The last Chapter IV is devoted to selected problems concerning the statistics of random fields with section headings: Linear forecasting for random fields observed on a sphere, On extrapolation of a homogeneous and isotropic random field from observation on a countable system of concentric spheres, Optimal estimates of regression coefficients and mean value of an isotropic random field observed on a sphere, On the integral equations for the statistics of homogeneous and/or isotropic random fields.

The bibliographical notes and a list of 276 references, 208 of which is to Soviet periodicals or books, are a very valuable help to the reader, especially to the Western reader.

Sándor Csörgő (Szeged)

Livres reçus par la rédaction

G. Alexits, Approximation theory. Selected papers. Edited by K. Tandori, 297 pages, Akadémiai Kiadó, Budapest, 1983.

V. I. Arnold, Geometrical methods in the theory of ordinary differential equations (Grundlehren der mathematischen Wissenschaften, Bd. 250), XI+334 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 94,—.

An Attribute Grammar for the Semantic Analysis of Ada. Edited by J. Uhl, S. Drossopoulou, G. Persch, G. Goos, M. Dausmann, G. Winterstein, W. Kirchgässner (Lecture Notes in Computer Science, Vol. 139), IX+511 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 52,—.

Complex Analysis, Methods, Trends, and Applications. Edited by E. Lanckau and W. Tutschke (Mathematische Lehrbücher und Monographien, II. Abteilung: Mathematische Monographien, Bd. 61), 398 pages, Akademie-Verlag, Berlin, 1983. — 78,— M.

Differential Equation Models. Edited by M. Braun, C. S. Coleman and D. A. Drew (Modules in Applied Mathematics, Vol. 1), XIX+380 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 69,—.

E. Fischer, Intermediate real analysis (Undergraduate Texts in Mathematics), XIV+770 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 86,—.

P. R. Halmos, Selecta expository writing. Edited by D. E. Sarason, L. Gillman, XIX+304 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 54,—.

——————, **Selecta research contributions.** Edited by D. E. Sarason, N. A. Friedman XXVIII+ +458 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 88,—.

Harmonic Maps. Proceedings of the NSF-CRMS Regional Conference, held at Tulane University, New Orleans, December 15—19, 1980. Edited by R. J. Knill, M. Kalka, H. C. J. Sealey (Lecture Notes in Mathematics, Vol. 949), V+158 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 24,—.

U. Kastens—B. Hutt—E. Zimmermann, Gag: A practical compiler generator (Lecture Notes in Computer Science, Vol. 141), IV+156 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 23,—.

S. Lang, Undergraduate analysis (Undergraduate Texts in Mathematics), XIII+545 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 94,—.

Life Science Models. Edited by H. Marcus-Roberts, M. Thompson (Modules in Applied Mathematics, Vol. 4), XX+366 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 72,—.

Mathematical Programming — The State of the Art, Bonn 1982. Edited by A. Bachem, M. Grötschel, B. Korte, VIII+655 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 128,—.

A. B. Mingarelli, Volterra-Stieltjes integral equations and generalized ordinary differential expressions

(Lecture Notes in Mathematics, Vol. 989), XIV+318 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 39,—.

Probability in Banach Spaces IV. Proceedings of the Seminar held in Oberwolfach, Germany, July 1982. Edited by A. Beck, K. Jacobs (Lecture Notes in Mathematics, Vol. 990), V+234 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 28,—.

Probability Theory and Mathematical Statistics. Proceedings of the Fourth USSR—Japan Symposium, held at Tbilisi, USSR, August 23—29, 1982. Edited by K. Itô, J. V. Prokhorov (Lecture Notes in Mathematics, Vol. 1021), VIII+747 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983. — DM 78,—.

Radical Banach Algebras and Automatic Continuity. Proceedings of a Conference held at California State University, Long Beach, July 17—31, 1981. Edited by J. M. Bachar, W. G. Bade, P. C. Curtis, H. G. Dales, M. P. Thomas (Lecture Notes in Mathematics, Vol. 975), VIII+470 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 55,—.

Séminaire de Probabilités XVII — 1981/82. Proceedings. Edité par J. Azéma, M. Yor (Lecture Notes in Mathematics, Vol. 986), V+512 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 62,—.

Stability Problems for Stochastic Models. Proceedings of the 6th International Seminar held in Moscow, USSR, April 1982. Edited by V. V. Kalashnikov and V. M. Zolotarev (Lecture Notes in Mathematics, Vol. 982), XVII+295 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983. — DM 39,—.

Studies in Pure Mathematics. To the Memory of Paul Turán. Editor-in-Chief: Paul Erdős, 773 pages, Akadémiai Kiadó, Budapest, 1983.

Twistor Geometry and Non-Linear Systems. Review Lectures given at the 4th Bulgarian Summer School on Mathematical Problems of Quantum Field Theory, held at Primorsko, Bulgaria, September 1980. Edited by H. D. Doebner and T. D. Palev (Lecture Notes in Mathematics, Vol. 970), V+216 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 28,—.

B. L. van der Waerden, **Geometry and algebra in ancient civilizations,** XII+223 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983. — DM 78,—.

M. I. Yadrenko, **Spectral theory of random fields** (Translation Series in Mathematics and Engineering), VII+259 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo and Optimization Software, Inc., New York, 1983. — DM 68,—.

A. C. Zaanen, **Riesz spaces II** (North-Holland Mathematical Library, Vol. 30), XI+720 pages, North-Holland Publ. Co., Amsterdam—Oxford—New York, 1983.

INDEX—TARTALOM

ACTA SCIENTIARUM MATHEMATICARUM

SZEGED (HUNGARIA), ARADI VÉRTANÚK TERE 1

On peut s'abonner à l'entreprise de commerce des livres et journaux „Kultúra" (1061 Budapest, I., Fő utca 32)

ISSN 034-6523 Acta Univ. Szeged
ISSN 0001-6969 Acta Sci. Math.

INDEX: 26024

84-323 — Szegedi Nyomda — F. v.: Dobó József igazgató

Felelős szerkesztő és kiadó: Leindler László
A kézirat a nyomdába érkezett: 1984. január 13
Megjelenés: 1984. augusztus

Példányszám: 1000. Terjedelem: 23,1 (A/5) ív
Készült monószedéssel, íves magasnyomással,
az MSZ 5601-24 és az MSZ 5602-55 szabvány szerint